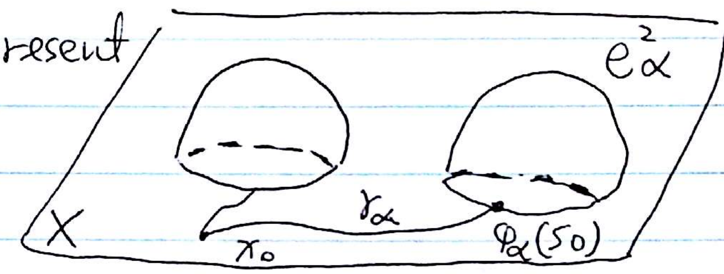
Applications to cell complexes

Let X be a path-connected space with $x_0 \in X$. Let Y be the space obtained by attaching 2-cells e_α^2 via the maps $\varphi_\alpha : S^1 \to X$.

We may consider φ_α as a loop in X with basepoint at $\varphi_\alpha(s_0)$ for some $s_0 \in S^1$, fixed for all α.

The loop φ_α does not represent an element of $\pi_1(X, x_0)$ unless $\varphi_\alpha(s_0) = x_0$



We choose a path $\gamma_\alpha : I \to X$ such that $\gamma_\alpha(0) = x_0$ and $\gamma_\alpha(1) = \varphi_\alpha(s_0)$, for each α.

Then $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ is a loop representing an element of $\pi_1(X, x_0)$.

(This loop may not be nullhomotopic in X but it will be nullhomotopic after e_α^2 is attached.)

Thus the normal subgroup $N \subset \pi_1(X, x_0)$ generated by all the loops $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ lies in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $X \hookrightarrow Y$.

Proposition 1.26

The inclusion $X \hookrightarrow Y$ induces a surjection

$$\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$$

whose kernel is N. Thus $\pi_1(Y) \simeq \pi_1(X)/N$.

Remark: N is independent of the choice of the paths γ_α.

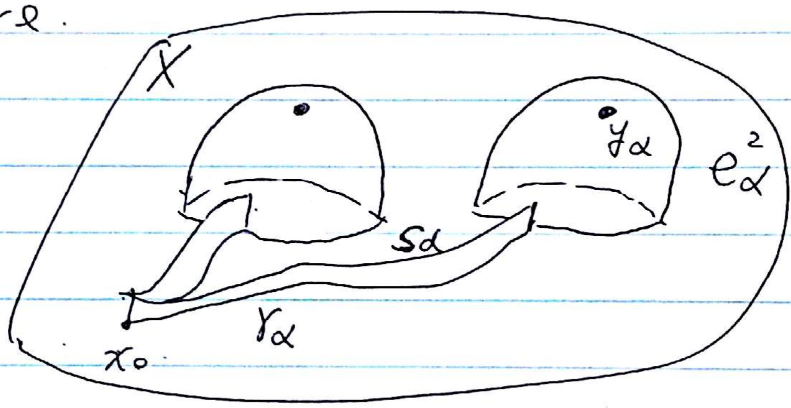
If η_α is another path from x_0 to $\varphi_\alpha(s_0)$, then

$$\eta_\alpha \varphi_\alpha \overline{\eta_\alpha} \simeq (\eta_\alpha \overline{\gamma_\alpha})(\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha})(\gamma_\alpha \overline{\eta_\alpha})$$
$$= (\eta_\alpha \overline{\gamma_\alpha})(\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha})\overline{(\eta_\alpha \overline{\gamma_\alpha})}$$

so $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ and $\eta_\alpha \varphi_\alpha \overline{\eta_\alpha}$ define conjugate elements of $\pi_1(X, x_0)$.

Proof of proposition:

For each path γ_α, we attach a band $S_\alpha = I \times I$ to Y as in figure.



Let $Z = Y \bigcup_\alpha S_\alpha$.

Then Y is a deformation retract of Z.

For each e^2_α, choose a point y_α in the interior away from S_α.

Let $A = Z - \bigcup_\alpha \{y_\alpha\}$

$B = Z - X$

Notice that X is a deformation retract of A. and that B is contractible.

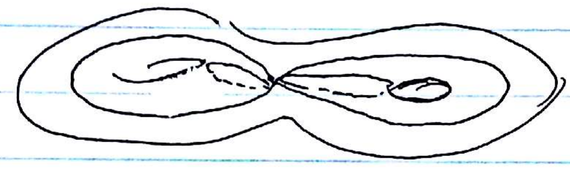
The intersection $A \cap B$ deformation retracts to the wedge of circles $\vee_\alpha S^1_\alpha$ and the image of $\pi_1(A \cap B) \to \pi_1(Z)$ generates the subgroup N.

Therefore $\pi_1(Y) \approx \pi_1(Z) \underset{\substack{\uparrow \\ \text{van Kampen's} \\ \text{theorem}}}{\approx} \dfrac{\pi_1(A) * \pi_1(B)}{N}$

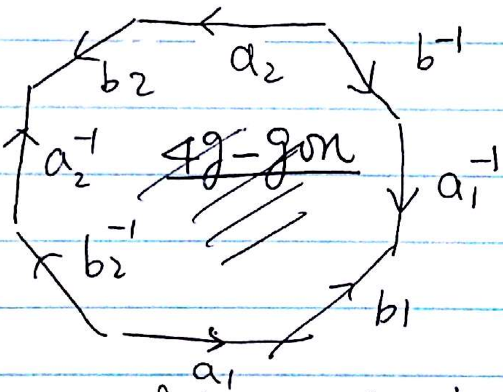
$\simeq \pi_1(X)/N.$ #

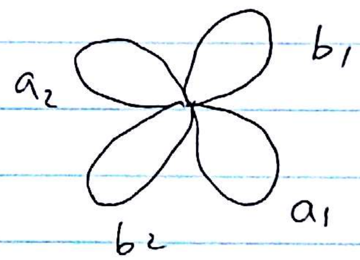
The closed orientable surface M_g of genus g.

M_3.



M_2





M_g is obtained by attaching one 2-cell to the wedge of $2g$ circles.

The attaching map $\varphi : S^1 \longrightarrow \underbrace{S^1 \vee \cdots \vee S^1}_{2g}$ is described by the word

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$

Therefore, by Proposition 1.26.

$$\pi_1(M_g) \approx \langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid [a_1, b_1][a_2, b_2]\cdots[a_g, b_g] \rangle$$

$$[a, b] = a b a^{-1} b^{-1}.$$

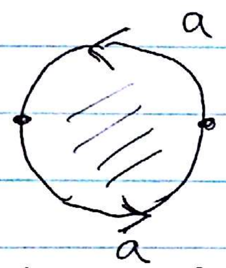
where $\langle g_\alpha | r_\alpha \rangle$ denotes the free group on the generators g_α modulo the normal subgroup generated by the words r_β in those generators.

<u>Corollary 1.2.?</u> The surface M_g is <u>not</u> homeomorphic or even homotopic equivalent to M_h if $g \neq h$.

pf:
The abelianization of $\pi_1(M_g)$ i.e. $\dfrac{\pi_1(M_g)}{[\pi_1(M_g), \pi_1(M_g)]}$ and $\pi_1(M_h)$ are free abelian groups with ranks $2g$ and $2h$, respectively. #

Nonorientable closed surfaces N_g.

N_1:

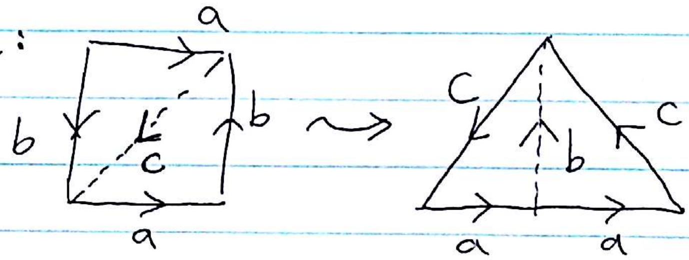


<u>projective plane</u>

RP^2.

the quotient of D^2 with antipodal points of ∂D^2 identified.

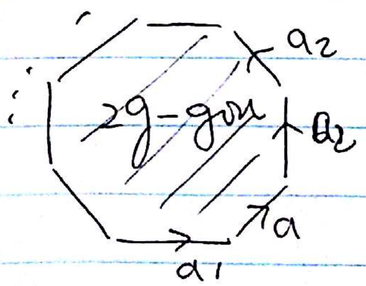
N_2:

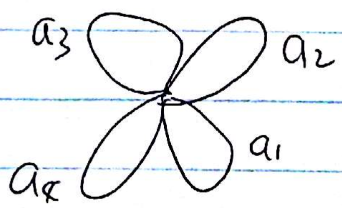


<u>Klein bottle</u>

a square with opposite sides identified via the word ~~$aba^{-1}b^{-1}$~~. $aba^{-1}b$

$\rightsquigarrow a^2c^2$.





A nonorientable surface N_g is obtained by attaching a 2-cell to the wedge sum of g circles by the word $a_1^2 \cdots a_g^2$.

By proposition 1.26, $\pi_1(N_g) \simeq \langle a_1, \cdots, a_g \mid a_1^2 \cdots a_g^2 \rangle$.

This abelianizes to $\mathbb{Z}_2 \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{g-1 \text{ copies}}$.

$\because$ We can choose generators to be

$a_1, \cdots, a_{g-1}, a_1 + a_2 + \cdots + a_g$ with $2(a_1 + a_2 + \cdots + a_g) = 0$.

Hence
- N_g is not homotopy equivalent to N_h if $g \neq h$
- N_g is not homotopy equivalent to any orientable surface M_h.

Corollary 1.28

For every group G, there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \simeq G$.

Pf: We can let $G = \langle g_\alpha \mid r_\alpha \rangle$, since every group is a quotient of a free group.

Construct X_G by attaching 2-cells e_β^2 to the wedge $\vee S_\alpha^1$ of circles via the attaching maps $\varphi_\beta : S^1 \longrightarrow \vee S_\alpha^1$ representing the word r_β.

※.

<u>Example 1.29</u> If $G = \langle a \mid a^n \rangle = \mathbb{Z}_n$.

X_G is obtained by attaching e^2 to S^1 via the map
$$\begin{array}{ccc} z & \mapsto & z^n \\ \cap & & \cap \\ S^1 & \longrightarrow & S^1 \end{array}$$

- $n=2$, $X_G = \mathbb{R}P^2$.

<u>Remark</u>: Attaching n-cells, $n \geq 3$ to a path-connected space has no effect on π_1, so all the interest lies in how the 2-cells are attached. cf. Exercise #6. on page 53.