Cellular Homology.

(Cellular homology is a very efficient tool for computing the homology groups of CW complexes, based on degree calculations.)

Lemma 2.34

If X is a CW complex, then

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$. and is free abelian for $k=n$, with a basis in one-to-one correspondence with n-cells of X.

(b). $H_k(X^n) = 0$ for $k > n$.

In particular, if X is finite-dimensional, then $H_k(X) = 0$ for $k > \dim X$.

(c) The inclusion $i: X^n \hookrightarrow X$ induces an isomorphism $i_*: H_k(X^n) \to H_k(X)$ if $k < n$.

Pf: (a).

Since (X^n, X^{n-1}) is a good pair, the quotient map $q: X^n \to X^n/X^{n-1}$ induces an isomorphism $q_*: H_k(X^n, X^{n-1}) \to \tilde{H}_k(X^n/X^{n-1})$ (prop. 2.22).

X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell e^n_α of X.

$$H_k(X^n, X^{n-1}) \approx \bigoplus_\alpha \tilde{H}_k(S^n) \approx \begin{cases} 0, & k \neq 0 \\ \bigoplus_\alpha \mathbb{Z}, & k = n. \end{cases}$$

(cor. 2.25).

(b) Consider the long exact sequence of (X^n, X^{n-1}).

$$\to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}) \to$$

$\forall k \neq n, n-1$, we have isomorphisms

$$H_k(X^{n-1}) \xrightarrow{\approx} H_k(X^n)$$

induced by inclusion map, by (a).

If $k > n$, then $H_k(X^n) \approx H_k(X^{n-1}) \approx \cdots \approx H_k(X^0) = 0$.

(c).

If $k < n$, then $H_k(X^n) \approx H_k(X^{n+1}) \approx \cdots \approx H_k(X^{n+m})$, $\forall m \geq 0$.

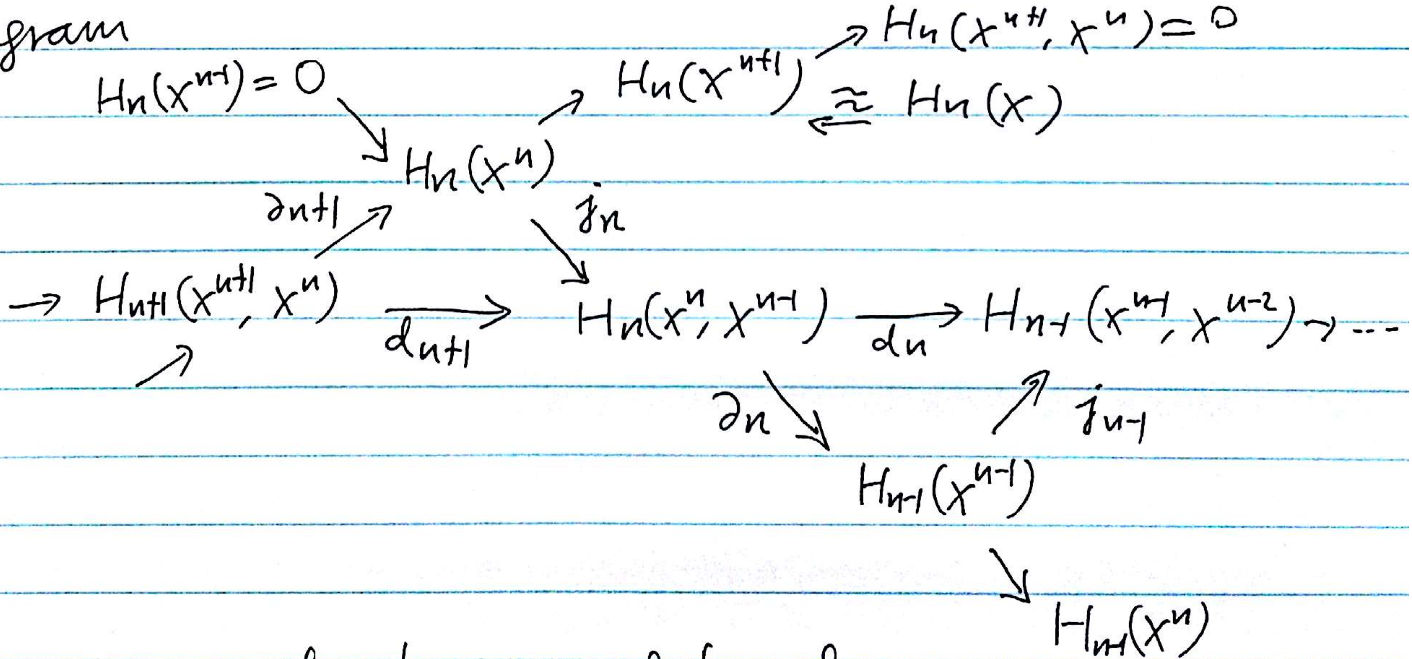
This proves (c) if X is finite-dimensional.

The case that X is of infinite dimensional, skip. #

Cellular chain complex.

Let X be a CW complex.

Using Lemma 2.34, portions of long exact sequences for (X^{n+1}, X^n), (X^n, X^{n-1}), (X^{n-1}, X^{n-2}) fit into a diagram



where d_{n+1} and d_n are defined as $j_n \partial_{n+1}$ and $j_{n-1}\partial_n$ respectively.

Then $d_n d_{n+1} = j_{n-1} \partial_n j_n \partial_{n+1} = 0$ $\because \partial_n j_n = 0$.

Thus the horizontal row in the diagram is a chain complex, called the <u>cellular chain complex</u> of X.

since $H_n(X^n, X^{n-1})$ is free with basis in 1-1 correspondence with the n-cells of X, so one can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n-cells of X.

The homology groups of ~~this~~ cellular chain complex are called the <u>cellular homology groups</u> of X.

Temporarily we denote them $H_n^{CW}(X) = \ker d_n / \operatorname{Im} d_{n+1}$.

<u>Theorem 2.35.</u> $H_n^{CW}(X) \approx H_n(X)$.

<u>pf:</u>

From the above diagram, $H_n(X) \approx H_n(X^n) / \operatorname{Im} \partial_{n+1}$.

Since j_n is injective, $\operatorname{Im} \partial_{n+1} \approx \operatorname{Im}(j_n \partial_{n+1}) = \operatorname{Im} d_{n+1}$

$H_n(X^n) \approx \operatorname{Im} j_n = \ker \partial_n$.

Since j_{n-1} is injective, $\ker \partial_n = \ker d_n$.

Thus j_n induces an isomorphism of $H_n(X^n) / \operatorname{Im} \partial_{n+1}$ onto $\ker d_n / \operatorname{Im} d_{n+1}$. #.

Applications of Theorem 2.35:

(i). $H_n(X) = 0$ if X has no n-cells.

(ii). If X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.

(iii). If X has neither $(n+1)$-cells nor $(n-1)$-cells, then $H_n(X)$ is free abelian with a basis in one-to-one correspondence with the n-cells of X.

$\because$ The cellular boundary maps d_n are 0.

$$H_n(X) \underset{\uparrow}{\approx} H_n^{CW}(X) \approx H_n(X^n, X^{n-1})$$

Thm 2.35.

$+$ Lemma 2.34 (a), the result follows.

Homology groups of $\mathbb{C}P^n$

$\mathbb{C}P^n$ has a CW structure, $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$.

By (iii), we have

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{for } i = 0, 2, 4, \cdots, 2n. \\ 0, & \text{otherwise.} \end{cases}$$

Cellular boundary formula

Let e_α^n, e_β^{n-1} denote the n-cells and $(n-1)$-cells of a CW complex X. The boundary map

$$d_n : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

satisfies

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

Example 2.36.

Let M_g be the closed orientable surface of genus g.
M_g consists of one 0-cell, $2g$ 1-cells, one 2-cell
The cellular chain complex of M_g is

$$0 \to H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \to 0.$$
$$\mathbb{Z} \qquad\qquad \mathbb{Z}^{2g} \qquad\qquad \mathbb{Z}$$

The attaching map of the 2-cell goes around each circle of the 1-skeleton twice in opposite directions.

Therefore the degree of the attaching map is 0 to each generator of $H_1(X^1, X^0)$, hence, $d_2 = 0$.

The boundary of each 1-cell is also 0.
Therefore we have the following chain complex.

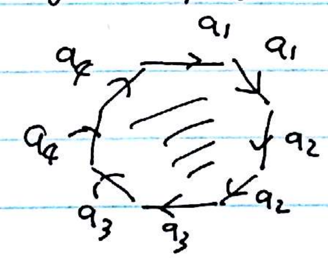
$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \to 0.$$

The homology groups of M_g are

$$H_i(M_g) = \begin{cases} \mathbb{Z}, & i = 0, 2 \\ \mathbb{Z}^{2g}, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.37.

Let N_g be the closed nonorientable surface of genus g.
N_g consists of 1 0-cell, g 1-cells, and 1 2-cell.



$g=4$.

The cellular chain complex is

$$0 \to H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \to 0$$

$$\mathbb{Z} \qquad\qquad \mathbb{Z}^g \qquad\qquad \mathbb{Z}$$

The attaching map of the 2-cell goes around each circle of X^1 twice in the same direction.
The boundary of each 1-ceel is 0.
Therefore $d_2(1) = (2, 2, \cdots, 2) \in \mathbb{Z}^g$ and $d_1 = 0$.

$\Rightarrow$ d_2 is injective. $\Rightarrow$ $H_2(N_g) = 0$.

If we change the basis for $\mathbb{Z}^g$ by replacing the last standard basis element $(0, \cdots, 0, 1)$ by $(1, 1, \cdots, 1)$

$\Rightarrow$ $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Therefore

$$H_i(N_g) = \begin{cases} \mathbb{Z}, & n=0. \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

These two examples illustrate the general fact that the orientability of a closed connected manifold M of dimension n is detected by $H_n(M)$.

$$H_n(M) = \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable.} \\ 0 & \text{if } M \text{ is nonorientable.} \end{cases}$$