<u>Recall: Cellular Boundary Formula.</u>

Let e^n_α, e^{n-1}_β denote the n-cells and $(n-1)$-cells of a CW complex X. The boundary map

$$d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

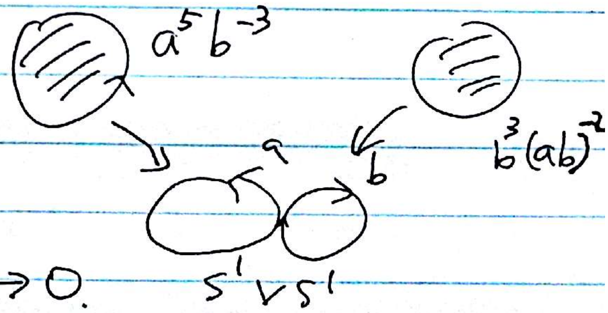
satisfies $$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the composition

$$S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$$ of the attaching map of e^n_α with the quotient map collapsing $X^{n-1} \setminus e^{n-1}_\beta$ to a point.

A space X is said to be <u>acyclic</u> if $\tilde{H}_i(X) = 0$ for all i.

<u>Example 2.38</u>. Let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words $a^5 b^{-3}$ and $b^3 (ab)^{-2}$



The cellular chain complex is $$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \to 0.$$

$d_1 = 0$, $d_1 : \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by the matrix $\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

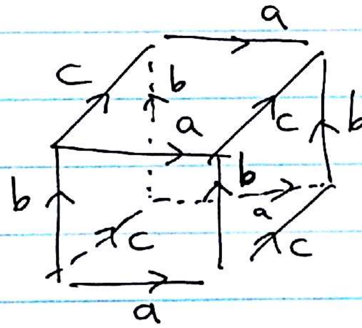
since $\begin{vmatrix} 5 & -2 \\ -3 & 1 \end{vmatrix} = -1$. d_2 has an inverse.

Hence d_2 is an isomorphism.

therefore $\widetilde{H}_i(X) = 0$ for all i.

<u>Example 2.39</u>

A 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ can be obtained by identifying opposite faces by $I \times I \times I$ as indicated in the figure.



The cellular chain complex is given by

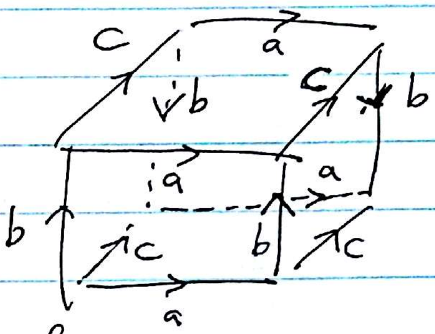
$$0 \to \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \to 0$$

On each of the three squares, the boundary of the 3-cell is mapped twice in opposite orientations. Therefore $d_3 = 0$.

Similarly, $d_2 = 0$ and $d_1 = 0$.

therefore $H_i(T^3) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^3 & i = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$

An identification of the opposite faces of the cube by the figure gives the product $K \times S^1$ where K is the Klein bottle.



We have a CW structure with 1 3-cell, 3 2-cells, 3 1-cells, 1 0-cell.

Thus the cellular chain complex have the form $0 \to \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$.

Use $\underset{(B,\,C)}{A}$ to denote the 2-cell given by the face orthogonal to the edge. a. (b, c).

$d_3 e^3 = 2C$, $d_2 A = 2b$, $d_2 B = 0$, $d_2 C = 0$.

$d_3(1) = (0, 0, 2)$. $\quad d_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\quad d_1 = 0$.

therefore
$$H_i(K \times S^1) = \begin{cases} \mathbb{Z} & i = 0. \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.42. Real projective space $\mathbb{R}P^n$.

$$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n.$$

The attaching map for e^k is the 2-sheeted covering projection $\varphi: S^{k-1} \to \mathbb{R}P^{k-1}$.

To compute the boundary map d_k we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1},$$

with q the quotient map.

$q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$.

These 2 homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$.

$$\Rightarrow \deg q\varphi = \deg \mathbb{1} + \deg(-\mathbb{1}) = 1 + (-1)^k.$$

$$d_k = \begin{cases} 0 & k: \text{odd} \\ \text{multiplication by } 2. & k: \text{even} \end{cases}$$

Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

if n is even

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

if n is odd.

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0,\ k = n \text{ odd}. \\ \mathbb{Z}_2 & k \text{ odd},\ 0 < k < n. \\ 0 & \text{otherwise}. \end{cases}$$