The fundamental group of the circle.

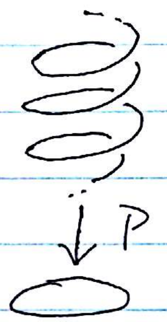
Theorem 1.7 The map $\Phi: \mathbb{Z} \to \pi_1(S^1)$ is an isomorphism.
$$n \mapsto [\omega_n].$$

$\omega_n: I \to S^1$ a loop based at $(1,0)$

$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$

Pf: Let $p: \mathbb{R} \to S^1$ be the map $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Let $\widetilde{\omega}_n: I \to \mathbb{R}$ a path in $\mathbb{R}$ from 0 to n such that $\widetilde{\omega}_n(s) = ns$



Then $\omega_n = p\widetilde{\omega}_n$

$\widetilde{\omega}_n$ is a lift of ω_n.

$\Phi(n) = [\omega_n] = [p\widetilde{\omega}_n]$

$$\begin{tikzcd} I \arrow[r, "\widetilde{\omega}_n"] \arrow[dr, "\omega_n"'] & \mathbb{R} \arrow[d, "p"] \\ & S^1 \end{tikzcd}$$

① Φ is a homomorphism

i.e. $\Phi(m+n) = \Phi(m) \cdot \Phi(n)$.

Pf of ①: Let $\tau_m: \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x+m$

Then $\widetilde{\omega}_m \cdot \tau_m \widetilde{\omega}_n$ is a path in $\mathbb{R}$ from 0 to $m+n$.

$$\widetilde{\omega}_m \cdot \tau_m \widetilde{\omega}_n(s) = \begin{cases} \widetilde{\omega}_m(2s), & 0 \le s \le \frac{1}{2} \\ m + \widetilde{\omega}_n(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

$$\Phi(m+n) = [p(\widetilde{\omega}_m \cdot \tau_m \widetilde{\omega}_n)] = [\omega_m \cdot \omega_n] = \Phi(m) \cdot \Phi(n)$$

② Φ is surjective.

Fact (a) (path lifting)

For each path $f: I \to S^1$ such that $f(0) = x_0 \in S^1$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f}: I \to \mathbb{R}$ such that $\widetilde{f}(0) = \widetilde{x}_0$.

pf of fact (a): Omit.

Pf of ②: given any loop $f: I \to S^1$.
By Fact (a), it can be lifted to a path $\tilde{f}: I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$.
Since f is a loop, we must have $\tilde{f}(1) \in \mathbb{Z}$.
Let $n = \tilde{f}(1)$. Then $\tilde{f} \simeq \tilde{\omega}_n$.
Hence $p\tilde{f} \simeq \omega_n$. Therefore $\Phi(n) = [\omega_n] = [p\tilde{f}] = [f]$.

③ Φ is injective

Fact (b) (Homotopy lifting)

For each homotopy $f_t: I \to S^1$ such that $f_t(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\tilde{f}_t: I \to \mathbb{R}$ such that $\tilde{f}_t(0) = \tilde{x}_0$.

Pf of fact (b): Omit.

Pf of ③: Suppose $\Phi(m) = \Phi(n)$. Then $\omega_m \simeq \omega_n$.
Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$.
By Fact (b), f_t lifts to a homotopy $\tilde{f}_t$ of paths starting at 0.
Uniqueness of (a) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$.
Since $\tilde{f}_t$ is a homotopy of paths $\tilde{f}_t(1)$ is independent of t.
$\tilde{f}_0(1) = \tilde{f}_1(1) \Rightarrow m = n$.

#

The following theorems can be proved using Theorem 1.7

Theorem 1.8. Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.
(Fundamental Theorem of Algebra)

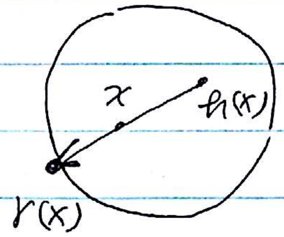
Pf: Omit.

<u>Theorem 1.9. (The Brouwer fixed point theorem in dim 2)</u>

Every continuous map $h: D^2 \to D^2$ has a fixed point.
i.e. a point x with $h(x) = x$.

<u>Pf</u>: Suppose $h(x) \neq x$, $\forall x \in D^2$.

Define a map $r: D^2 \to S^1$ by taking $x \in D^2$ to the point on the ray from $h(x)$ to x that intersects S^1.



r is continuous. ($\because$ small perturbations of x
$\Rightarrow$ " " " of $h(x)$
$\Rightarrow$ " " " of $r(x)$.)

If $x \in S^1$, $r(x) = x$. i.e. $r|_{S^1} = id_{S^1}$

r is a retraction of D^2 onto S^1.

We will show that no such retraction exists.

Let f_0 be any loop in S^1.

Since D^2 is convex.

The linear homotopy $f_t(s) = (1-t) f_0(s) + t x_0$,
where x_0 is the basepoint of f_0.
is a homotopy of f_0 to a constant loop at x_0 in D^2.

Then $r \circ f_t$ is a homotopy in S^1
from $r \circ f_0 = f_0$ to the constant loop at x_0.
($\because r|_{S^1} = id_{S^1}$).

Hence $\pi_1(S^1) = 0$.

But $\pi_1(S^1) \neq 0$. $\rightarrow\leftarrow$.

#.

Theorem 1.10 (Borsuk-Ulam theorem in dim 2.)
For every continuous map $f: S^2 \to \mathbb{R}^2$, there exists a pair of antipodal points x and $-x$ in S^2, with $f(x) = f(-x)$.

Remarks

① There exists a pair of antipodal points on the surface of the earth having the same temperature and the same barometric pressure at every single moment.

② There is no one-to-one map from S^2 into $\mathbb{R}^2$.

pf of theorem 1.10.

Suppose $f(x) \neq f(-x)$ for every $x \in S^2$.

Define a map $g: S^2 \to S^1$ by $g(x) = \dfrac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$.

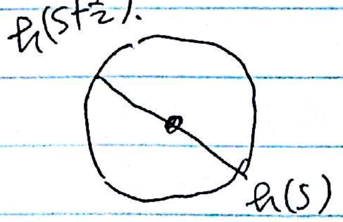
Let $\eta: I \to S^2$ be the loop of the equator

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0).$$

and let $h: I \to S^1$ be the composite $h = g\eta$.

Since $g(-x) = \dfrac{f(-x) - f(x)}{\| f(-x) - f(x) \|} = -g(x)$,

we obtain
$$h(s+\tfrac{1}{2}) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0)$$
$$= g(-\eta(s)) = -g\eta(s) = -h(s)$$
for $s \in [0, \frac{1}{2}]$.



(This means $h(s)$ and $h(s+\frac{1}{2})$ differ by an odd number of half rotation around S^1.)

We ~~lift~~ lift the loop h to a path $\tilde{h}: I \to \mathbb{R}$.

$h(s+\frac{1}{2}) = -h(s)$ implies that $\tilde{h}(s+\frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$
for some odd integer q.

Since q depends continuously on s and has integer values, it is a constant.

In particular, $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$.

This means that h goes around S^1 q times.

Since q is odd, h represents a nontrivial element in $\pi_1(S^1)$.

But $h = g\eta : I \to S^2 \to S^1$ and η is nullhomotopic in S^2.

so $g\eta$ is nullhomotopic in S^1

h reprensents the trivial element in $\pi_1(S^1)$.

$\Rightarrow \Leftarrow$.

Corollary 1.11 When S^2 is expressed as the union of three closed sets A_1, A_2 and A_3, then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$.

Pf: Let $d_i : S^2 \to \mathbb{R}$ measur distance to A_i.

i.e. $d_i(x) = \inf_{y \in A_i} |x - y|$

This is a continuous function.

Apply Borsuk-Ulam theorem to the map $S^2 \to \mathbb{R}^2$
$x \mapsto (d_1(x), d_2(x))$

obtaining a pair of antipodal points x and $-x$.
with $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$.

If $d_i(x) = d_i(-x) = 0$ for some i, then $\{x, -x\} \subseteq A_i$ ($\because A_i$ closed)

If $d_i(x) = d_i(-x) \neq 0$ for $i = 1, 2$, then $\{x, -x\} \cap (A_1 \cup A_2) = \emptyset$.

$\Rightarrow \{x, -x\} \subset A_3$.

Proposition 1.12 $\pi_1(X\times Y) \overset{\text{isom.}}{\approx} \pi_1(X)\times\pi_1(Y)$ if X, Y are path-connected.

Pf: A basic property of the product topology:
A map $f: Z \to X\times Y$ is continuous iff $g: Z\to X$, $h: Z \to Y$ given by $f(z)=(g(z), h(z))$, $z\in Z$ are continuous.

Hence a loop f in $X\times Y$ based at (x_0, y_0) is equivalent to a pair of loops g in X based at x_0 and h in Y based at y_0.
Similarly, a homotopy f_t of a loop in $X\times Y$ is equivalent to a pair of homotopies of the loops g_t in X and h_t in Y.

$\Rightarrow$ We obtain a bijection $\pi_1(X\times Y, (x_0, y_0))$

$$\approx \pi_1(X, x_0)\times \pi_1(Y, y_0)$$
$$[f] \mapsto ([g], [h]).$$

This is obviously a group homomorphism, and hence an isomorphism. #.

Example 1.13 The torus.

$$\pi_1(S^1\times S^1) \approx \pi_1(S^1)\times\pi_1(S^1) \approx \mathbb{Z}\times\mathbb{Z}.$$

$$\pi_1(\underbrace{S^1\times S^1\times\cdots\times S^1}_{k \text{ copies}}) \approx \underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k \text{ copies}}.$$