§1.3 Covering Spaces

A covering space of a space X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$ satisfying:

There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α by p.

(The empty disjoint union is allowed, so p need not be surjective.)

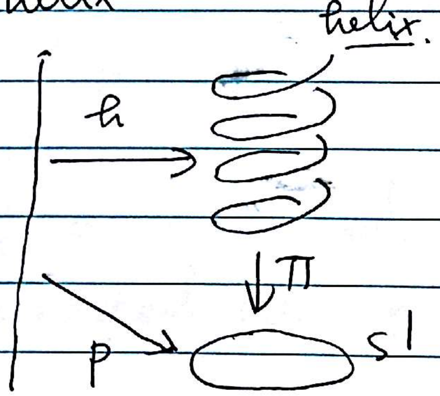
Examples: (1). Let $p: \mathbb{R} \to S^1$ be given by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

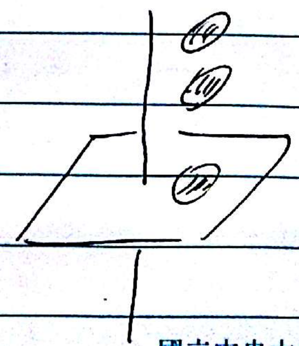
ex: $\{U_\alpha\}$: any two open arcs whose union is S^1.

Notice that p factors through the helix:

$$\begin{tikzcd} t \arrow[r, mapsto, "h"] \arrow[dr, "p"'] & (\cos 2\pi t, \sin 2\pi t, t) \arrow[d, "\pi"] \\ & (\cos 2\pi t, \sin 2\pi t) \end{tikzcd}$$



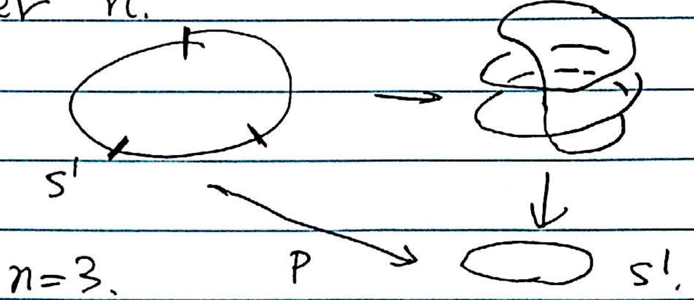
(2). The helicoid surface $S = \{(s\cos 2\pi t, s\sin 2\pi t, t) \mid 0 < s < a,\ t \in \mathbb{R}\}$ projects onto the punctured plane $\mathbb{R}^2 - \{0\}$ via the map $(x, y, z) \mapsto (x, y)$.



The preimage of any open disk U of $\mathbb{R}^2 - \{0\}$ is a disjoint union of countably many open disk on S, each mapped homeomorphically onto S.

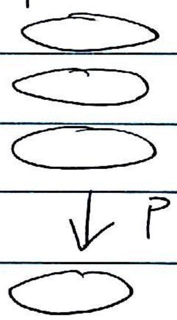
(3) The map $p: S^1 \to S^1$ given by $z \mapsto z^n$ $|z|=1$, $z \in \mathbb{C}$
for some positive integer n.

p maps around the image n times.



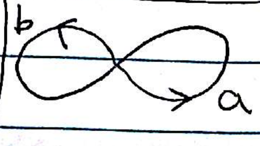
• As our general theory will show, (1) and (3) are the only connected covering spaces of S^1!

(4) Covering spaces which are disconnected.



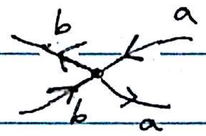
• A disconnected covering space of a connected space is a disjoint union of connected covering space.

(5). The covering spaces of $S^1 \vee S^1$.
Set $X = S^1 \vee S^1$.
We view X as a graph with one vertex and two edges a and b.
We choose orientations for a and b.



Let $\tilde{X}$ be any other graph with 4 edges meeting at each vertex.

Suppose the edges of $\tilde{X}$ have been assigned labels a and b and orientations in such a way that the local picture near each vertex is the same as X.

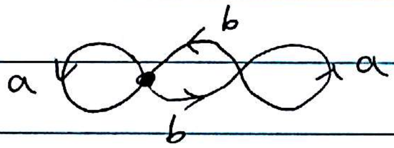


We call $\tilde{X}$ a <u>2-oriented graph.</u>

~~Covering spaces of $X \longleftrightarrow$ graphs that inherits a 2-orientation from X.~~

See page 58 for some covering spaces of $S^1 \vee S^1$!

Ex:



$\langle a, b^2, bab^{-1} \rangle$.

<u>Lifting properties.</u>

<u>Recall:</u> <u>A lift</u> of a map $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ such that $p\tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

<u>Homotopy lifting property (covering homotopy property)</u>

<u>Prop 1.30:</u> Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$ and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Ex: $p: \mathbb{R} \to S^1$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[ur, "\exists!\, \tilde{f}_t"] \arrow[r, "f_t"'] & S^1 \end{tikzcd}$$

$\tilde{f}_0$: a lift of f_0.

- Take Y to be a point, we get the path lifting property for a covering space $p: \tilde{X} \to X$.

$$\begin{tikzcd} & \tilde{X} \ni \tilde{x}_0 \arrow[d, "p"] \\ I \arrow[ur, "\exists!\, \tilde{f} \text{ s.t. } \tilde{f}(0)=\tilde{x}_0"] \arrow[r, "f"'] & X \ni x_0 \end{tikzcd}$$

$\tilde{x}_0$: a lift of x_0.

$f(0) = x_0$.

- Take Y to be I, every homotopy f_t of a path f_0 in X lifts to a homotopy $\tilde{f}_t$ of each lift $\tilde{f}_0$ of f_0.

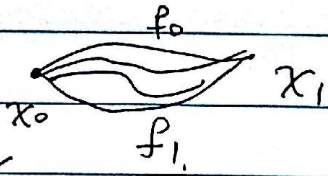
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "\exists \text{ lifted homotopy } \tilde{f}_t"] \arrow[r, "f_t"'] & X \end{tikzcd}$$

$\tilde{f}_0$: a lift of f_0.

- The lifted homotopy $\tilde{f}_t$ is a homotopy of paths, fixing the endpoints by ① and ②.

① The endpoints of the homotopy f_t are fixed: i.e. they are constant paths.

② The uniqueness of lifts in path lifting property for $p: \tilde{X} \to X$ implies that every lift of a constant path is constant



Prop 1.31 (1) The map $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by a covering space $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is injective.

(2) The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Pf: (1)

- An element of the kernel of p_* is represented by a loop $\tilde{f}_0: I \longrightarrow \tilde{X}$ with a homotopy $f_t: I \to X$ of $f_0 = p\tilde{f}_0$ to the trivial loop f_1.
- Remarks above implies that there is a lifted homotopy of loops $\tilde{f}_t$ starting with $\tilde{f}_0$ and ending with a constant loop.
- Hence $[\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{x}_0)$.

$\Rightarrow$ p_* is injective. #

(2). Elements of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are of the form

$p_*([\tilde{f}]) = [p\tilde{f}]$ for some loop $\tilde{f}$ based at $\tilde{x}_0$

- Let f be a loop based at x_0 s.t. $p\tilde{f} \simeq f$.
- The homotopy describing this, say f_t, lifts to a homotopy $\tilde{f}_t$ s.t. $\tilde{f}_0 = \tilde{f}$.
- Since $f_t(1) = x_0$, $\tilde{f}_t(1) \in p^{-1}(x_0)$ for all t.

$t \mapsto \tilde{f}_t(1)$ is a continuous map into the discrete space $p^{-1}(x_0)$.
- As I is connected, this map must be constant.

$\Rightarrow \tilde{f}_1(1) = \tilde{f}_0(1) = \tilde{f}_0(0) = \tilde{x}_0$. #

If $p: \tilde{X} \to X$ is a covering space,
The argument that $t \mapsto \tilde{f}_t(1)$ is constant shows that the cardinality of $p^{-1}(x)$ is locally constant over X.
Hence, if X is connected, this cardinality is constant as x ranges over X.
This number is called the number of <u>sheets</u> of the covering.

<u>Prop 1.32</u> The number of sheets of a covering space $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

<u>pf:</u>

Prop 1.30. in the case $Y = I$ and $f_t: I \to X$ is a homotopy of loops f_0 and f_1 based at x_0.

$\Rightarrow$ The lifts $\tilde{f}_0$ and $\tilde{f}_1$ of f_0 and f_1 starting at $\tilde{x}_0$ end the same point.

$\Rightarrow$ There is a well-defined map
$$\phi: \pi_1(X, x_0) \to p^{-1}(x_0).$$
defined by $\phi[f] = \tilde{f}(1)$, where $\tilde{f}$ is the lift of f starting at $\tilde{x}_0$.

Let $\tilde{x}_1 \in p^{-1}(x_0)$.
Since $\tilde{X}$ is path-connected, there is a path $\tilde{f}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. $\Rightarrow$ $\phi[p\tilde{f}] = \tilde{f}(1) = \tilde{x}_1$.
This shows that ϕ is **surjective**.

Suppose that g_1, g_2 are loops in X based at x_0 such that $\phi[g_1] = \phi[g_2]$.
Let $\tilde{g}_1, \tilde{g}_2$ be lifts of g_1, g_2, respectively both starting at $\tilde{x}_0$.
Then $\tilde{g}_1(1) = \phi[g_1] = \phi[g_2] = \tilde{g}_2(1)$.
Therefore $\tilde{g}_1 \cdot \overline{\tilde{g}_2}$ is a loop based at $\tilde{x}_0$.
Since $p_*[\tilde{g}_1 \cdot \overline{\tilde{g}_2}] = [p\tilde{g}_1 \cdot p\overline{\tilde{g}_2}] = [g_1][\overline{g_2}] = [g_1][g_2]^{-1}$
we have the equality $H[g_1] = H[g_2]$,

where $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Therefore ϕ induces a bijective map

$$\Phi : \pi_1(X, x_0)/H \longrightarrow p^{-1}(x_0)$$

defined by $\Phi(H[f]) = \phi[f] = \tilde{f}(1)$.

Therefore the index of H in $\pi_1(X, x_0)$ equals the number of sheets of $p : \tilde{X} \to X$. #

<u>Lifting criterion</u>

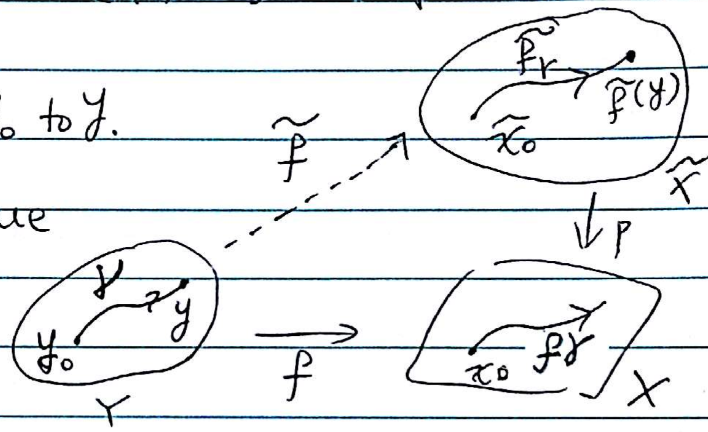
<u>Prop 1.33</u> Given a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \longrightarrow (X, x_0)$ with Y path-connected and locally path-connected, a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Recall: A space X is <u>locally path-connected</u> if <u>for every</u> $x \in X$ and <u>each</u> neighborhood U of x, there <u>is an open neighborhood</u> $V \subset U$ of x that is path-connected.

Pf of prop 1.33

If there is a lift $\tilde{f}$, then $f_*(\pi_1(Y,y_0)) = p_* \tilde{f}_*(\pi_1(Y,y_0))$
$\subseteq p_* \pi_1(\tilde{X},\tilde{x}_0)$.

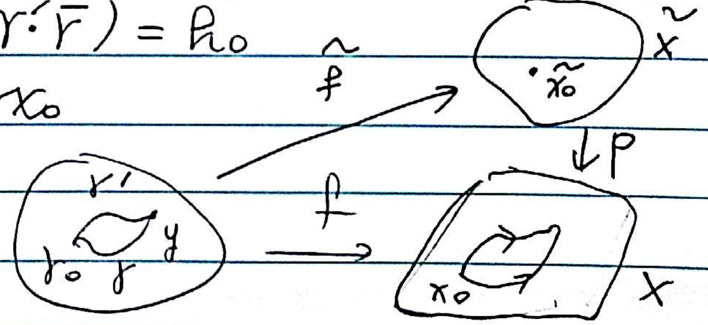
Conversely, suppose that $f_*(\pi_1(Y,y_0)) \subseteq p_* \pi_1(\tilde{X},\tilde{x}_0)$.
We construct $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ as follows:
given $y\in Y$, there is a path γ in Y from y_0 to y.
The path $f\gamma$ has a unique lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$.



Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

$\tilde{f}$ is well-defined:

If γ' is another path in Y from y_0 to y,
then $(f\gamma')\cdot\overline{(f\gamma)} = f(\gamma'\cdot\bar{\gamma}) = h_0$
is a loop based at x_0
satisfying $[h_0] = f_*[\gamma'\cdot\bar{\gamma}]$
$\in f_*(\pi_1(Y,y_0))$
$\subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$.



This means that h_0 is homotopic to a loop h_1 which lifts to a loop $\tilde{h}_1$ based at $\tilde{x}_0$.
By the covering homotopy property, the homotopy h_t between h_0 and h_1 lifts to a homotopy $\tilde{h}_t$ in $\tilde{X}$.
Since $\tilde{h}_1$ is a loop based at $\tilde{x}_0$, so is $\tilde{h}_0$.

- The first half of $\widetilde{h_0}$ is a lift of $f\gamma'$ starting at $\tilde{x}_0$ and the second half is a lift of $\overline{f\gamma}$ ending at $\tilde{x}_0$.
- Therefore $\tilde{h}_0(\frac{1}{2})$ is the common endpoint of the lifts of $f\gamma'$ and $f\gamma$ starting at $\tilde{x}_0$ showing that $\tilde{f}$ is well-defined.

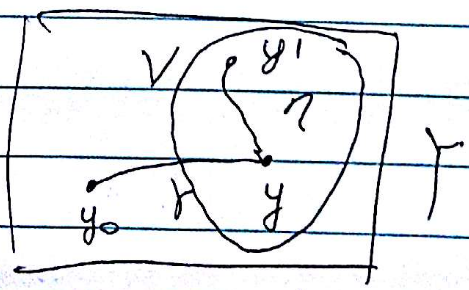
$\tilde{f}$ is continuous:

Let $U \subset X$ be an open neighborhood of $f(y)$ such that $\tilde{U} \subset \tilde{X}$ is a sheet of $p^{-1}(U)$ containing $\tilde{f}(y)$ which is mapped homeomorphically onto U by p.

Let V be a path-connected neighborhood of y such that $f(V) \subset U$. Such a V exists since f is continuous and Y is locally path-connected.

For paths from y_0 to points $y' \in V$ we can take a fixed path γ from y_0 to y followed by paths η in V from y to y'.

Then the paths $(f\gamma)\cdot(f\eta)$ in X have lifts $(\widetilde{f\gamma})\cdot(\widetilde{f\eta})$, where $\widetilde{f\eta} = p^{-1}f\eta$ and $p^{-1}: U \to \tilde{U}$ is the inverse of $p: \tilde{U} \to U$.



Thus $\tilde{f}(V) \subset \tilde{U}$ and $\tilde{f}|_V = p^{-1}f$, hence $\tilde{f}$ is continuous at y.

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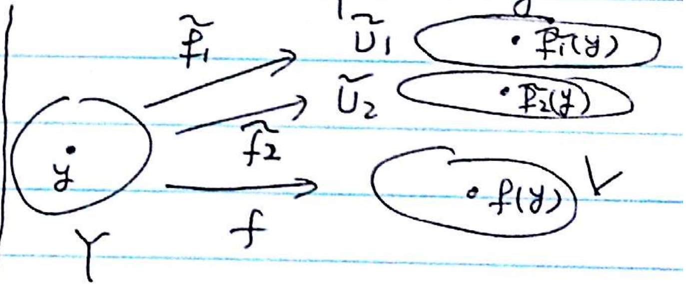
Prop 1.34 (Unique lifting property)

Given a covering space $p: \widetilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ of f agree at one point of Y and if Y is connected, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on all of Y.

Pf:

For each point $y \in Y$, let U be an open neighborhood of $f(y)$ in X for which $p^{-1}(U)$ is a disjoint union of open sets $\widetilde{U}_\alpha$ each mapped homeomorphic to U by p, and let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets of $p^{-1}(U)$ containing $\widetilde{f}_1(y)$, $\widetilde{f}_2(y)$ respectively.

By continuity of $\widetilde{f}_1$ and $\widetilde{f}_2$, there is a neighborhood N of y mapped into $\widetilde{U}_1$ by $\widetilde{f}_1$ and into $\widetilde{U}_2$ by $\widetilde{f}_2$.



If $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then $\widetilde{U}_1 \neq \widetilde{U}_2$, hence $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$ and $\widetilde{f}_1 \neq \widetilde{f}_2$ on N.

On the other hand, if $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then $\widetilde{U}_1 = \widetilde{U}_2$ so $\widetilde{f}_1 = \widetilde{f}_2$ on N since $p\widetilde{f}_1 = p\widetilde{f}_2$ and p is injective on $\widetilde{U}_1 = \widetilde{U}_2$.

Thus the set of points where $\widetilde{f}_1$ and $\widetilde{f}_2$ agree is both open and closed in Y.

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