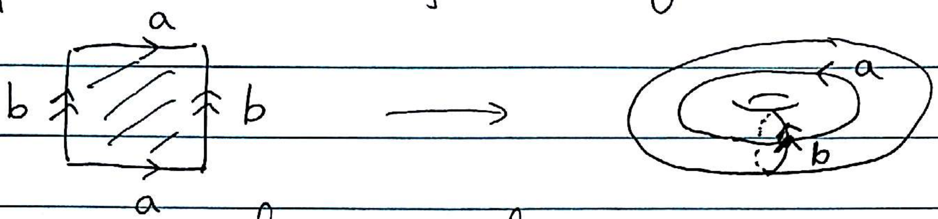
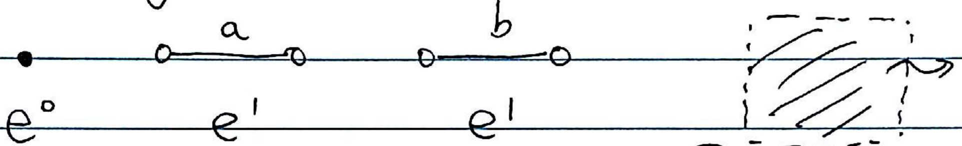
Cell complexes

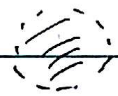
A torus $S^1 \times S^1$ can be constructed by identifying opposite sides of a square.



On the other hand, a torus can be constructed by using



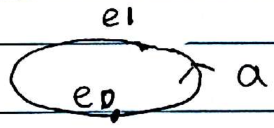
e^0 ① $\quad$ e^1 ② $\quad$ e^1 ③ $\quad$ ④ e^n can be thought as an open disk.

Def: e^n: an n-cell, homeomorphic to the open n-disk $D^n - \partial D^n$ (ex:  : $D^2 - \partial D^2$)

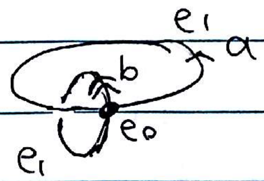
D^0; e^0: consist of a single point.

$S^0 = \partial D^1$ consists of two points.

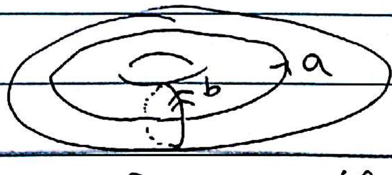
① + ② $\Rightarrow$



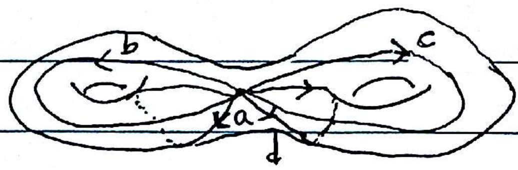
① + ② + ③ $\Rightarrow$

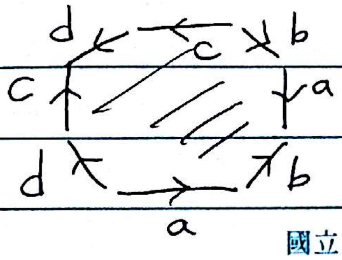


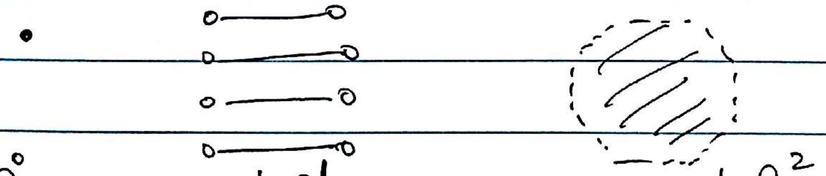
① + ② + ③ + ④ $\Longrightarrow$



Similarly, the closed surface with genus 2 can be constructed in a similar way.







$1e^0$ $\quad$ $4e^1$ $\quad$ $1e^2$

$4g$-edges $\Rightarrow$ $2g$ circles $\Rightarrow$ g genus

Def: A space is called a <u>cell complex</u> or a <u>CW complex</u> if it is the union of an increasing sequence $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots$ of subspaces in the following way:

(1) X^0 is a discrete set whose points are regarded as 0-cells.

(2). Inductively, form the <u>n-skeleton</u> X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$.

i.e. X^n is the quotient space of the disjoint union $X^n = X^{n-1} \amalg_\alpha D^n_\alpha \Big/ x \sim \varphi_\alpha(x)\,,\ x \in \partial D^n_\alpha$.

Thus, as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$, where e^n_α is an open n-disk.

(3). One can either (a) stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$.

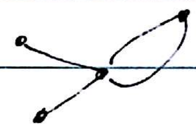
or (b) one can continue indefinitely, setting $X = \bigcup_n X^n$

In case (b), X is given the <u>weak topology</u>:

$A \subset X$ is open (closed) iff $A \cap X^n$ is open (closed) in X^n, $\forall n$

- If $X = X^n$ for some n, then X is said to be <u>finite dimensional</u> & the smallest such n is the <u>dimension</u> of X.
- If X is finite dimensional, then the maximum dimension of cells of X is the dimension of X.

<u>Example 0.1</u>: 1-dimensional cell complexes are called graphs.



<u>Example 0.2</u>: skipped.

<u>Example 0.3</u>: The sphere S^n.

(1) $S^n = e^0 \cup e^n$.

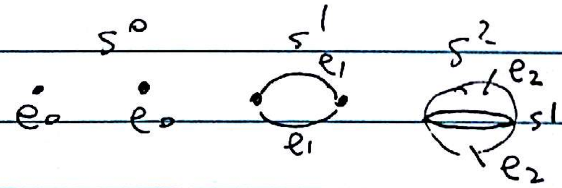
$X^0 = e^0,\ X^1 = X^2 = \cdots = X^{n-1},\ X^n = e^0 \cup e^n$.

$\varphi : S^{n-1} \longrightarrow X^0$ constant map.

$S^n =$ "one-point compactification of e^n"

(2) $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^n \subseteq S^{n+1} \subseteq \cdots\cdots$

obtained from S^{n-1} by attaching 2 n-cells.



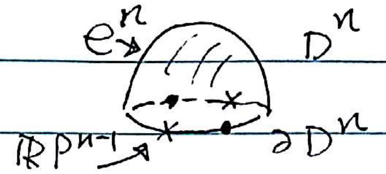
$$S^\infty = \bigcup_{n=0}^{\infty} S^n.$$

<u>Example 0.4</u>: The real projective n-space $\mathbb{R}P^n$.

$$\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\} \Big/ \underset{\lambda \in \mathbb{R}\setminus\{0\}}{v \sim \lambda v} = S^n \Big/ \underset{v \in S^n}{v \sim -v} = D^n \Big/ \underset{v \in \partial D^n = S^{n-1}}{v \sim -v}$$

∂D^n with antipodal points identified is $\mathbb{R}P^{n-1}$.

$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$.



$\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$

$X^k = e^0 \cup e^1 \cup \cdots \cup e^k = \mathbb{R}P^{k-1}$, $\quad k = 0, 1, \cdots, n$.

$\varphi_k : S^{k-1} \longrightarrow X^{k-1} = \mathbb{R}P^{k-1}$. $\quad \varphi_k(x) = \varphi_k(-x)$.

<u>Example 0.5</u> : $\mathbb{R}P^\infty$.

$$\mathbb{R}P^0 \subseteq \mathbb{R}P^1 \subseteq \cdots \subseteq \mathbb{R}P^n \subseteq \cdots \qquad \mathbb{R}P^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}P^n$$

<u>Example 0.6</u>: The complex projective space $\mathbb{C}P^n$.

$$\mathbb{C}P^n = \mathbb{C}^{n+1} - \{0\} \Big/ \underset{\lambda \in \mathbb{C},\ \lambda \neq 0.}{v \sim \lambda v} \qquad v = (z_0 \cdots z_n) \in \mathbb{C}^{n+1} - \{0\}.$$

$$= S^{2n+1} \big/ v \sim \lambda v, \quad \lambda \in \mathbb{C},\ |\lambda| = 1.$$

Let $D_+^{2n} = \{ (\omega, \sqrt{1 - |\omega|^2}) \in \mathbb{C}^n \times \mathbb{C} \mid |\omega| \le 1 \} \subset S^{2n+1}$

$$\partial D_+^{2n} = \{ (\omega, 0) \in \mathbb{C}^n \times \mathbb{C} \mid |\omega| = 1 \} = S^{2n-1}.$$

$$\mathbb{C}P^n = D_+^{2n} \big/ v \sim \lambda v, \quad \underset{(z_0, \cdots, z_{n-1}, 0)}{\overset{}{v \in \underset{\parallel}{S^{2n-1}}}}, \quad \lambda \in \mathbb{C},\ |\lambda| = 1.$$

$\Rightarrow$ $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$.

By induction on n,

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

Similarly, $\mathbb{C}P^\infty$ has a cell structure with one cell in each even dimension.

- If $\phi_\alpha : S_\alpha^{n-1} \to X^{n-1}$ the attaching map of e_α^n, then we define the corresponding characteristic map

$$\Phi_\alpha : D_\alpha^n \to X$$

as the composite $D_\alpha^n \hookrightarrow X^{n-1} \amalg_\alpha D_\alpha^n \to X^n \hookrightarrow X$.
where the middle map is the quotient map defining X.

Example 0.3 $\Rightarrow$ $S^n = e^0 \cup e^n = D^n / \partial D^n$.
A characteristic map for e^n is the quotient map $D^n \to S^n$ collapsing ∂D^n to a point.

Example 0.4 $\Rightarrow$ $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$.
A characteristic map for e^i is the quotient map $D^i \to \mathbb{R}P^i \subset \mathbb{R}P^n$ identifying antipodal points of ∂D^i.
similarly for $\mathbb{C}P^n$.

Def: A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X.

$\because$ A is closed, the image of the characteristic map of each cell in A is contained in A.
So A is a cell complex in its own right.

- If A is a subcomplex of a cell complex X, then the pair (X, A) is called a CW-pair.

EX: (1) Each skeleton X^n of a cell complex X.
is a subcomplex
(X^n, X) is a CW-pair.
(2) $\mathbb{RP}^k$, $k \le n$ is a subcomplex of $\mathbb{RP}^n$.
(3). $\mathbb{CP}^k$, $k \le n$. is a subcomplex of $\mathbb{CP}^n$.

- If S^n is given the cell structure $e^0 \cup e^n$, then the subspheres $S^0, S^1, \cdots, S^{n-1}$ are not subcomplexes.

- But if S^n is given the cell structure.
$$S^k = S^{k-1} \cup 2\ k\text{-cells}, \quad k = 0, 1, \cdots, n$$
then S^i, $i < n$ is a subcomplex of S^n.

Remark: In the above examples, the closure of a single cell is a subcomplex.
But there are cases that the closure of a single cell is not a subcomplex.

Consider cases that the attaching map φ_α is not onto a cell.