§2.2 Computations and applications

Degree

For a map $f: S^n \to S^n$ with $n>0$, the induced map $f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \to \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}}$, $f_*(\alpha) = d\alpha$ for some integer d and a generator $\alpha \in H_n(S^n)$.

Let $k\alpha \in H_n(S^n)$ be an arbitrary element, then $f_*(k\alpha) = f_*(\alpha + \cdots + \alpha) = f_*(\alpha) + \cdots + f_*(\alpha) = k(d\alpha) = d(k\alpha)$

$\Rightarrow$ d depends only on f.

The integer d is called the <u>degree</u> of f, $\deg f$.

properties of degree.

(a) $\deg \mathbb{1} = 1$, since $\mathbb{1}_* = \mathbb{1}$.

(b). $\deg f = 0$ if f is <u>not</u> surjective.

For if we choose a point $x_0 \in S^n - f(S^n)$, then f can be factored as a decomposition

$$S^n \to S^n - \{x_0\} \hookrightarrow S^n$$

Since $S^n - \{x_0\}$ is contractible, $H_n(S^n - \{x_0\}) = 0$.

Hence $f_* = 0$.

(c). If $f \simeq g$, then $\deg f = \deg g$ since $f_* = g_*$.

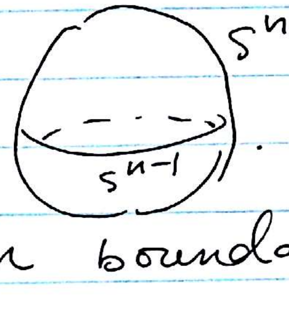
The converse statement, that $f \simeq g$ if $\deg f = \deg g$ is a fundamental theorem of Hopf around 1925.

(d). $\deg fg = \deg f \cdot \deg g$ since $(fg)_* = f_* g_*$.

As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $fg \simeq \mathbb{1}$. $\Rightarrow \deg f \cdot \deg g = \deg \mathbb{1} = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n about a subsphere S^{n-1}.

Consider S^n as the quotient of $\Delta_1^n \cup \Delta_2^n$ with $\partial\Delta_1^n = \partial\Delta_2^n$,



the reflection about the common boundary interchanges Δ_1^n and Δ_2^n.

Therefore $f_*(\underbrace{\Delta_1^n - \Delta_2^n}) = \Delta_2^n - \Delta_1^n = (-1)(\Delta_1^n - \Delta_2^n)$.

($\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ Example 2.23.)

(f). The antipodal map $-\mathbb{1}: S^n \to S^n$, $x \to -x$ has degree $(-1)^{n+1}$.

since it is the composition of $(n+1)$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.

(g). If $f: S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.

For if $f(x) \neq x$, then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \le t \le 1$, does not pass through the origin.

Hence if f has no fixed points, then $f_t(x) = \dfrac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$ defines a homotopy from f to the antipodal map.

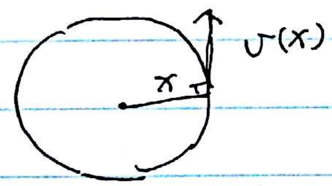
By (c) and (f), we have $\deg f = \deg(-\mathbb{1}) = (-1)^{n+1}$.

Theorem 2.28

S^n has a continuous field of nonzero tangent vectors iff n is odd.

Pf:

Suppose $x \mapsto v(x)$ is a tangent vector field on S^n, then $x \cdot v(x) = 0$



$n=1$.

If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x.

Then $f_t(x) = \cos t \cdot x + \sin t \cdot v(x)$, for $0 \le t \le \pi$, $\underset{S^n}{\in}$

is a homotopy from id map of S^n to the antipodal map $-\mathbb{1}$.

$\Rightarrow \deg(-\mathbb{1}) = \deg \mathbb{1}$, hence $(-1)^{n+1} = 1$ and n must be odd.

Conversely, if n is odd, say $n = 2k-1$,

we can define $v(x_1, x_2, \cdots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$

Then $v(x)$ is orthogonal to x,

so v is a tangent vector fied on S^n,

and $|v(x)| = 1$ for all $x \in S^n$.

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Corollary: A tangent vector field on S^{2n} vanishes somewhere.

Recall that an <u>action</u> of a group G on a space X is a homomorphism from G to the group $\text{Homeo}(X)$ of homeomorphisms $X \to X$.

The action is <u>free</u> if the homeomorphism corresponding to each nontrivial element of G has no fixed points..

In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$.

<u>Prop. 2.29</u>. $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.

<u>pf:</u> since the degree of a homeomorphism must be ± 1,
an action of G on S^n determines a degree function $d: G \to \{\pm 1\}$.
This is a homomorphism since $\deg fg = \deg f \cdot \deg g$
if the action is free, then d sends every nontrivial element of G to $(-1)^{n+1}$ by (g).
Thus when n is even, d has trivial kernel, (1–1).
so $G \subset \mathbb{Z}_2$.