Computation of degrees.

Suppose $f: S^n \to S^n$, $n > 0$ has the property that for some point $y \in S^n$, $f^{-1}(y)$ consists of finitely many points, say $x_1, \dots, x_m$.

Let $U_1, \dots, U_m$ be disjoint neighborhoods of $x_1, \dots, x_m$, mapped by f into a neighborhood V of y.

Then $f(U_i - x_i) \subset V - y$, $\forall i$.

We have the following commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\simeq"'] \arrow[d, "k_i"] & H_n(V, V-y) \arrow[d, "\simeq"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\simeq"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\simeq"']
\end{tikzcd}$$

where k_i and p_i are induced by inclusions, the two isomorphisms in the upper half of the diagram come from excision, the lower two isomorphism come from exact sequences of pairs.

$\Rightarrow$ $H_n(U_i, U_i - x_i) \simeq H_n(S^n) \simeq \mathbb{Z}$, $H_n(V, V-y) \simeq H_n(S^n, S^n - y)$
$\simeq H_n(S^n) \simeq \mathbb{Z}$.

The top homomorphism f_* becomes multiplication by an integer, called the <u>local degree</u> of f at x_i, written $\deg f|x_i$.

prop 2.30 $\deg f = \sum_i \deg f|x_i$.

pf: By excision, $H_n(S^n, S^n - f^{-1}(y)) \approx \bigoplus_{i=1}^{m} \underbrace{H_n(U_i, U_i - x)}_{\substack{\text{s.i.}\\ \mathbb{Z}}}$.

with k_i the inclusion of the ith summand

P_i the projection onto the ith summand.

The commutativity of the lower triangle says that $P_i j(1) = 1$, hence $j(1) = (1, \cdots, 1) = \sum_i k_i(1)$

The commutativity of the upper square says that the middle f_* takes $k_i(1)$ to $\deg f|x_i$.

hence $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f|x_i$.

The commutativity of the lower square gives that $\deg f = \sum_i \deg f|x_i$.

#

Example 2.31 (We can use prop. 2.30 to construct a map $S^n \to S^n$ of any given degree, $\forall n \geq 1$.)

Let $q: S^n \longrightarrow \bigvee_k S^n$ be the quotient map obtained by collapsing the complement of k disjoint open balls B_i in S^n to a point.

Let $p: \bigvee_k S^n \longrightarrow S^n$ identify all the summands to a single sphere.

Consider $f = pq$.

For almost all $y \in S^n$, $f^{-1}(y)$ consisting of one point x_i in each B_i.

The local degree of f at x_i is ± 1 since f is a homeomorphism near x_i.

By precomposing p with reflections of the summands of $\bigvee_k S^n$ if necessary, we can make each local degree $+1$ or -1.

Thus we can produce a map $S^n \to S^n$ of degree $\pm k$

Example 2.32 In the case of S^1, the map $f(z) = z^k$ has degree k.

if $k=0$, then f = constant.

The case $k<0$ reduces to the case $k>0$ by composing with $z \mapsto z^{-1}$, which is a reflection, of degree -1.

For $k>0$, every point in S^1 has k points in the preimage.

At each of these k points, the local degree is 1

Prop 2.33. $\deg Sf = \deg f$, where $Sf: S^{n+1} \to S^{n+1}$ is the suspension of the map $f: S^n \to S^n$.

Pf:

Let CS^n denote the cone $S^n \times I / S^n \times 1$ with base $S^n = S^n \times 0 \subset CS^n$, so CS^n / S^n is the suspension of S^n.

The map f induces $Cf: (CS^n, S^n) \to (CS^n, S^n)$ with quotient Sf. $\left(Sf(x,t) = (f(x), t) \right)$.

Recall: $SS^n = S^{n+1}$.

consider the long exact sequence of (CS^n, S^n)

$$\to \underset{\overset{\|}{0}}{\tilde{H}_{n+1}(CS^n)} \to H_{n+1}(CS^n, S^n) \xrightarrow[\cong]{\partial} \tilde{H}_n(S^n) \to \underset{\overset{\|}{0}}{\tilde{H}_n(CS^n)}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} H_{n+1}(SS^n) & \xleftarrow{\cong} & H_{n+1}(CS^n, S^n) & \xrightarrow[\cong]{\partial} & \tilde{H}_n(S^n) \\ (Sf)_* \downarrow & & (Cf)_* \downarrow & & \downarrow f_* \\ H_{n+1}(SS^n) & \xleftarrow{\cong} & H_{n+1}(CS^n, S^n) & \xrightarrow[\cong]{\partial} & \tilde{H}_n(S^n) \end{array}$$

Since the two squares commute, we conclude that $\deg Sf = \deg f$. #.