Chapter 2. Homology.

§2.1 Simplicial and singular homology

<u>n-simplex</u> : n-dim'l analogy of the triangle.

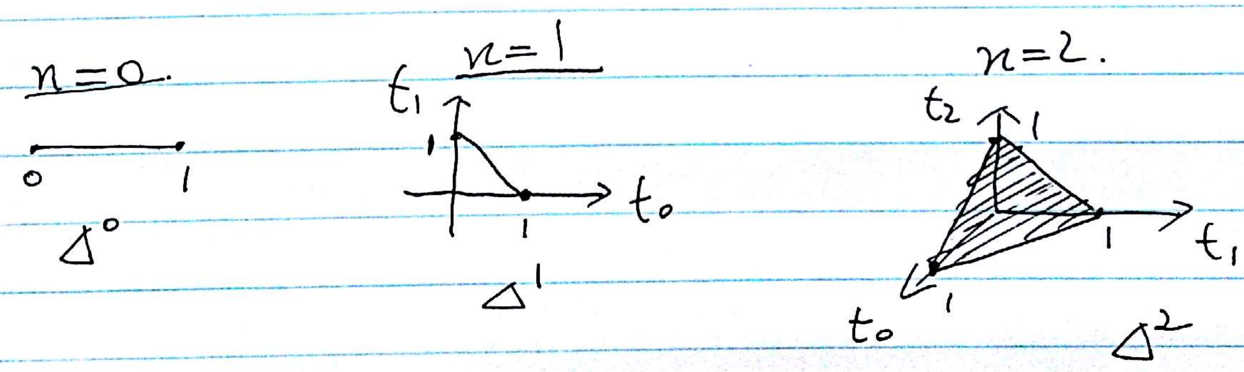
- Let $v_0, v_1, \dots, v_n$ be points of $\mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent.

The <u>n-simplex</u>, denoted by $[v_0, v_1 \cdots, v_n]$, is the smallest convex set of $\mathbb{R}^m$ containing $v_0, v_1, \dots, v_n$.

- $v_0, v_1, \dots, v_n$ are called the <u>vertices</u> of the simplex.
- The n-simplex $[v_0, v_1, \dots, v_n]$ does not lie in a hyperplane of dimension less than n.

<u>Example:</u> $\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1,\ t_i \geq 0,\ \forall i\}$

the <u>standard n-simplex.</u>



- "n-simplex" will really mean "<u>n-simplex with an ordering of its vertices.</u>"

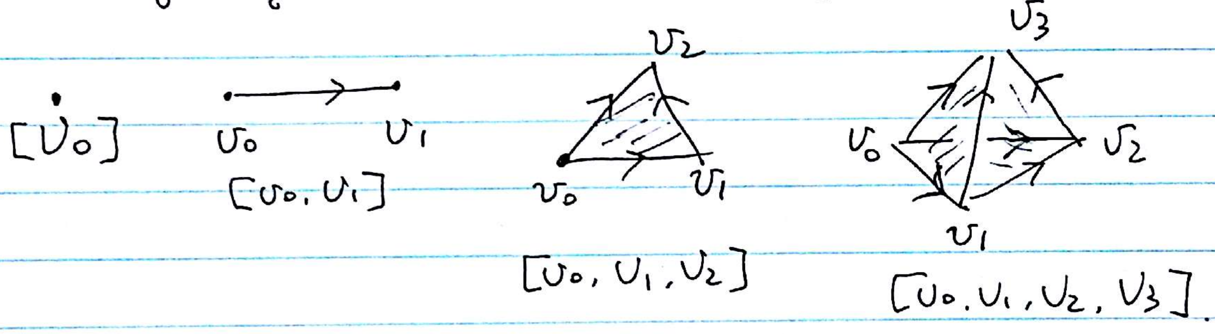
An ordering of the vertices of $[v_0, v_1, \dots, v_n]$ determines orientations of the edges $[v_i, v_j]$.

Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, \cdots, v_n]$

$$\underset{\Delta^n}{\underset{\cap}{(t_0, \cdots, t_n)}} \longmapsto \underset{[v_0, \cdots, v_n]}{\underset{\cap}{\sum_i t_i v_i}}, \quad \sum t_i = 1,\ t_i \geq 0.$$

The coefficients t_i are called the <u>barycentric coordinates</u> of $\sum_i t_i v_i$ in $[v_0, \cdots v_n]$.

Example:



If we delete one of the vertices of $[v_0, \cdots, v_n]$, then the remaining n vertices span an $(n-1)$-simplex, called <u>a face</u> of $[v_0, \cdots, v_n]$.

The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the <u>boundary</u> of Δ^n, "$\partial\Delta^n$." The <u>open simplex</u> $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.

A <u>Δ-complex structure</u> on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i). The restriction $\sigma_\alpha|\mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\mathring{\Delta}^n$.

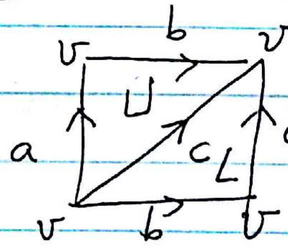
(ii). Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \longrightarrow X$.
Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear isomorphism between them that preserves the ordering of the vertices.

(iii). A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

(iii) $\Rightarrow$ $\sigma_\alpha|\mathring{\Delta}^n$ is an homeomorphism onto its image, an open simplex in X

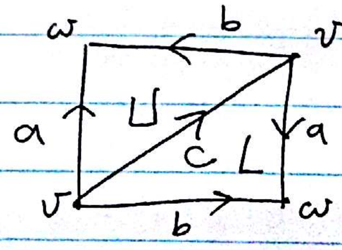
<u>Example:</u>

<u>Torus</u> $T = S^1 \times S^1$



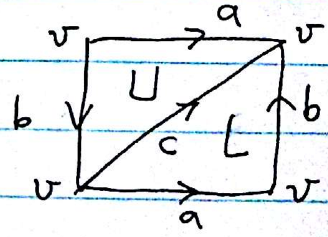
6 σ_α's

<u>The real projective plane.</u> $\mathbb{R}P^2$

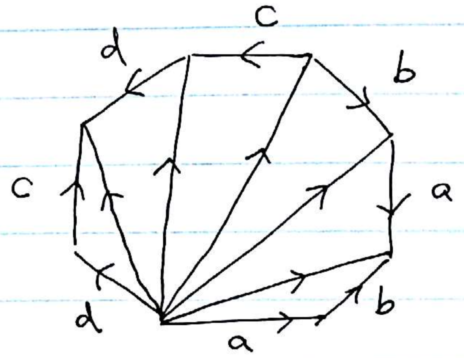


7 σ_α's

<u>Klein bottle</u> K.



6 σ_α's.



The closed orientable surface with genus 2.

X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha : \Delta^n \to X$, by identifying each face of a Δ^n_α with. with the Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face in question.

One can think of building the quotient space inductively, starting with a discrete set of vertices, then attaching edges to these to produce a graph, then attaching 2-simplices to the graph, and so on.

a Δ-complex can be described purely combinatorically as collections of n-simplices Δ^n_α for each n together with functions associating to each face of each n-simplex Δ^n_α an $(n-1)$-simplex Δ^{n-1}_β.

a Δ-simplex X is a CW-complex whose n-cells are the open simplices $\sigma_\alpha(\mathring{\Delta}^n)$'s with σ_α's as the characteristic maps.