Euler characteristic

For a finite CW complex X, the Euler characteristic $\chi(X)$ is the alternating sum $\sum_n (-1)^n C_n$, where C_n is # of i-cells in X.

Ex: For a 2-dim'l CW complexes X, $\chi(X) = V - E + F$.

Theorem 2.44. $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$.

Theorem 2.44 $\Rightarrow$ $\chi(X)$ can be defined purely in terms of homology. and, hence depends only on the homotopy type of X.

In particular, $\chi(X)$ is independent of the choice of CW structure on X.

Recall: The rank of a finitely generated abelian group G is the number of $\mathbb{Z}$-summands of G.

if $0 \to A \to B \to C \to 0$ is an exact sequence of finitely generated abelian groups, then

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Pf of 2.44:

Let $0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$. be a chain complex of finitely generated abelian groups, with cycles $Z_n = \ker d_n$,

boundary $B_n = \operatorname{Im} d_{n+1}$.

homology $H_n = Z_n / B_n$.

Thus we have short exact sequences

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

and $\quad 0 \to B_n \to Z_n \to H_n \to 0,$

hence $\quad \text{rank } C_n = \text{rank } Z_n + \text{rank } B_{n-1}$

$\quad \text{rank } Z_n = \text{rank } B_n + \text{rank } H_n.$

$$\begin{aligned}\sum_i (-1)^i \text{ rank } H_i &= \sum_i (-1)^i (\text{rank } Z_i - \text{rank } B_i) \\ &= \sum_i (-1)^i (\text{rank } C_i - \text{rank } B_{i-1} - \text{rank } B_i) \\ &= \sum_i (-1)^i \text{ rank } C_i - \cancel{\sum_i (-1)^i \text{ rank } B_{i-1} - \sum_i (-1)^i \text{ rank } B_i} \\ &= \sum_i (-1)^i \text{ rank } C_i.\end{aligned}$$

Applying this with $C_n = H_n(X^n, X^{n-1})$ then gives the theorem. #

<u>split exact sequence</u>

Suppose one has a retraction $r: X \to A$, so $ri = \mathbb{1}$, where $i: A \to X$ is the inclusion.

The induced map $i_*: H_n(A) \to H_n(X)$ is then injective since $r_* i_* = \mathbb{1}$.

Consider the long exact sequence

$$H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \underset{r_*}{\overset{i_*}{\rightleftarrows}} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to$$

$$\Rightarrow \quad 0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \to 0.$$

$(\because i_* \text{ is injective}, \ \text{Im } \partial = \ker i_* = 0).$

$r_* i_* = \mathbb{1}$ gives more information.

<u>Splitting Lemma.</u> For a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ of abelian groups the following are equivalent:

(a). There exists $p : B \to A$ such that $pi = \mathbb{1} : A \to A$.

(b). There exists $s : C \to B$ such that $js = \mathbb{1} : C \to C$.

(c). There is an isomorphism $B \approx A \oplus C$ making

$$\begin{tikzcd} & & B \arrow[dr, "j"] \arrow[d, "\approx"] & & \\ 0 \arrow[r] & A \arrow[ur, "i"] \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

$$a \mapsto (a,0), \qquad (a,c) \mapsto c$$

commutative.

- If these conditions are satisfied, the exact sequence is said to <u>split</u>.

<u>Sketch of proof:</u>

For (a) $\Rightarrow$ (c), one checks that the map $B \to A \oplus C$, $b \mapsto (p(b), j(b))$ is an isomorphism with the desired property.

For (b) $\Rightarrow$ (c), one uses the map $A \oplus C \to B$, $(a,c) \mapsto i(a) + s(c)$.

(c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b) are obvious.

If one wants to show (b) $\Rightarrow$ (c) directly, one can define $p(b) = i^{-1}(b - sj(b))$.

... Exercise. #

Recall that we have

$$0 \to H_n(A) \underset{r_*}{\overset{i_*}{\rightleftarrows}} H_n(X) \to H_n(X,A) \to 0$$

$r_* i_* = \mathbb{1}$.

By the splitting Lemma,

$$H_n(X) \approx H_n(A) \oplus H_n(X,A).$$

As an application, we know that for $n \geq 2$.

S^{n-1} is <u>not</u> a retract of D^n

Since a retraction $D^n \to S^{n-1}$ will give a splitting

$$\underset{\substack{\wr\wr \\ 0}}{H_{n-1}(D^n)} \approx \underset{\substack{\wr\wr \\ \mathbb{Z}.}}{H_{n-1}(S^{n-1})} \oplus H_{n-1}(D^n, S^{n-1})$$

$\rightarrow\leftarrow$.