Exact sequence.

- A sequence of homomorphisms
$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \rightarrow \cdots$$
is said to be <u>exact</u> if $\operatorname{Ker}\alpha_n = \operatorname{Im}\alpha_{n+1}$, $\forall n$.

① $\operatorname{Im}\alpha_{n+1} \subset \ker\alpha_n \iff \alpha_n\alpha_{n+1} = 0$.

so the sequence is a chain complex.

② $\ker\alpha_n \subset \operatorname{Im}\alpha_{n+1} \Rightarrow H_n = \dfrac{\ker\alpha_n}{\operatorname{Im}\alpha_{n+1}} = 0$.

Therefore an exact sequence is a chain complex with trivial homology groups.

- An exact sequence of the form
$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$
is called a <u>short exact sequence.</u>

Basic properties:

(i) $0 \rightarrow A \xrightarrow{\alpha} B$ is exact. $\iff \operatorname{Ker}\alpha = 0$.
i.e. α is injective

(ii) $A \xrightarrow{\alpha} B \rightarrow 0$ is exact. $\iff \operatorname{Im}\alpha = B$
i.e. α is surjective.

(iii). $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow 0$. is exact. $\iff$ α is an isomorphism

(iv). $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is exact.

$\iff$ α is injective, β is surjective, $\ker\beta = \operatorname{Im}\alpha$.

so β induces an isomorphism $C \simeq B/\operatorname{Im}\alpha$

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