Excision

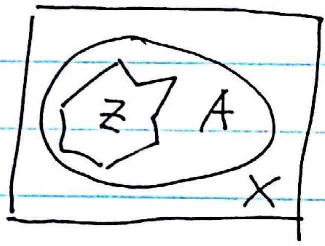
Excision Theorem describes when $H_n(X, A)$ are unaffected by deleting or excising a subset $Z \subset A$.

Theorem 2.20

Given subspaces $Z \subset A \subset X$ such that $\underline{\overset{\mathrm{cl}\,Z}{\overset{\|}{\bar{Z}}} \subset \mathrm{int}(A)}$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X-Z, A-Z) \to H_n(X, A)$ for all n.

Equivalently, for subspaces $A, B \subset X$ such that $\underline{\mathrm{int}(A) \cup \mathrm{int}(B) = X}$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$, $\forall n$.

- Set $B = X - Z$ and $Z = X - B$.
  Then $A \cap B = A - Z$ and
  the condition $\underline{\mathrm{cl}\,Z \subset \mathrm{int}\,A}$
  $\iff \underline{X = \mathrm{int}\,A \cup \mathrm{int}\,B}$
  $\because X - \mathrm{int}\,B = \mathrm{cl}\,Z \subset \mathrm{int}\,A$



Theorem 2.13

If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to$$
$$\xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \cdots \to \tilde{H}_0(X/A) \to 0$$

where $i: A \hookrightarrow X$ is the inclusion and $j: X \to X/A$ is the quotient map.

Pairs of (X, A) satisfying the hypothesis of the theorem will be called good pairs.

Theorem 2.13 follows from the following proposition.

prop 2.22

For good pairs (X, A), the quotient map $q: (X, A) \to (X/A, A/A)$ induces isomorphisms $q_*: H_n(X, A) \to H_n(X/A, A/A) \approx \tilde{H}_n(X/A)$ for all n.

Pf: Let V be a neighborhood of A in X that deformation retracts onto A.

We have a commutative diagram

$$\begin{array}{ccccc} H_n(X,A) & \to & H_n(X,V) & \to & H_n(X-A, V-A) \\ \downarrow q & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \to & H_n(X/A, V/A) & \to & H_n(X/A - A/A, V/A - A/A) \end{array}$$

- A deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$ and $H_n(A, A) = 0. \Rightarrow H_n(V, A) = 0.$

Then the long exact sequence

$$\to H_n(V, A) \to H_n(X, A) \to H_n(X, V) \to H_{n-1}(V, A) \to$$

gives isomorphisms $H_n(X, A) \xrightarrow{\approx} H_n(X, V)$ for all n

$\Rightarrow$ The upper left horizontal map is an isomorphism

- The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism.
- The other two horizontal maps are isomorphisms directly from excision.
- The right-hand vertical map q_* is an isom. since q restricts to a homeomorphism on the complement of A.
- From the commutativity of the diagram $\Rightarrow$ the left-hand q_* is an isomorphism. #.

Cor. 2.14

$\tilde{H}_n(S^n) \approx \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$.

Pf: For $n > 0$ take $(X, A) = (D^n, S^{n-1})$

so $X/A = S^n$.

We have the long exact sequence.

$$\to \tilde{H}_i(D^n) \to \tilde{H}_i(D^n/S^{n-1}) \to \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n) \to$$

Since D^n is contractible, $\tilde{H}_i(D^n) = 0$.

Exactness of the sequence $\Rightarrow \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and $\tilde{H}_0(S^n) = 0$.

$\tilde{H}_0(S^0) \approx \mathbb{Z}$ and $\tilde{H}_i(S^0) = 0$, $i > 0$.

The result follows by induction on n. #.

<u>Cor 2.24</u>. If X is a CW complex and if A, B are subcomplexes such that $A\cup B = X$, $A\cap B \neq \emptyset$, then the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$ for all n.

<u>Pf</u>: Since CW pairs are good, prop 2.22 allows us to pass to $B/A\cap B$ and X/A which are homeomorphic. #

<u>Cor 2.25</u>. For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\approx} \tilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed at basepoint $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

<u>Pf</u>:

$$\tilde{H}_n(X_\alpha) \approx H_n(X_\alpha, x_\alpha). \quad \forall n.$$

The proof follows from prop. 2.22 by taking

$$(X, A) = \left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\right).$$

#

<u>Example</u>: $X = S^1 \vee S^1 \vee \cdots \vee S^1$ the wedge sum of k circles.

$$\tilde{H}_n(S^1 \vee \cdots \vee S^1) \approx \bigoplus_{i=1}^{k} \tilde{H}_n(S^1) = \begin{cases} \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{k\text{-copies}}, & n=1 \\ 0, & \text{otherwise.} \end{cases}$$

Invariance of domain.

Theorem 2.26 If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Pf:

For $x \in U$, we have $H_k(U, U-\{x\}) \approx H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by excision.

From the long exact sequence for $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$, we get $H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \xrightarrow{\cong} \tilde{H}_{k-1}(\mathbb{R}^m - \{x\})$.

Since $\mathbb{R}^m - \{x\}$ deformation retracts onto S^{m-1}, we conclude that $H_k(U, U-\{x\})$ is $\mathbb{Z}$ for $k = m$ and 0 otherwise.

By the same reasoning $H_k(V, V-\{y\})$ is $\mathbb{Z}$ for $k = n$ and 0 otherwise.

Since a homeomorphism $h: U \to V$ induces isomorphisms $H_k(U, U-\{x\}) \to H_k(V, V-\{h(x)\})$, $\forall k$. we must have $m = n$.

#

- $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m \neq n$.

Naturality

For a map $f:(X,A)\to(Y,B)$, the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) & \to \cdots \\
& \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
\cdots \to & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y,B) & \xrightarrow{\partial} & H_{n-1}(B) & \to \cdots
\end{array}$$

is commutative.

Commutativity of the squares involving i_* and j_* are obvious as they are induced by commutative diagrams of corresponding squares of chain groups.

We only need to check $f_*\partial = \partial f_*$.

For a class $[\alpha]\in H_n(X,A)$ represented by a relative cycle α, we have

$$f_*\partial[\alpha] = f_*[\partial\alpha] = [f_\#\partial\alpha] = [\partial f_\#\alpha]$$
$$= \partial[f_\#\alpha] = \partial f_*[\alpha].$$