Homotopy and homotopy type.

Algebraic topology: uses "algebraic tools" to study "topological spaces."

Goal: Find algebraic invariants that classify topological spaces "up to homeomorphism", mostly classify "up to homotopy equivalence".

Chapter 0. Some underlying geometric notions.

spaces: topological spaces.

maps: continuous maps.

Defi: A deformation retraction of a space X onto a subspace A is a family of maps

$$f_t : X \to X, \quad t \in I := [0, 1].$$

such that $f_0 = \mathbb{1}$ (the identity map)

$$f_1(X) = A.$$

$$f_t|_A = \mathbb{1} \quad \forall t \in I.$$

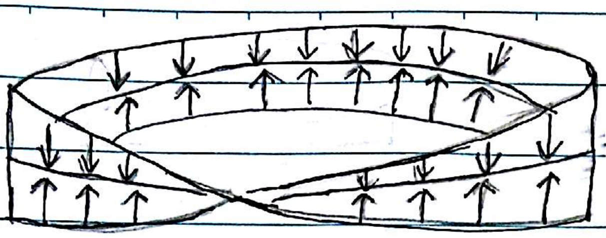
- The family $f_t(x) = F(x, t)$ is continuous

  i.e. $F : X \times I \to X$ is continuous.

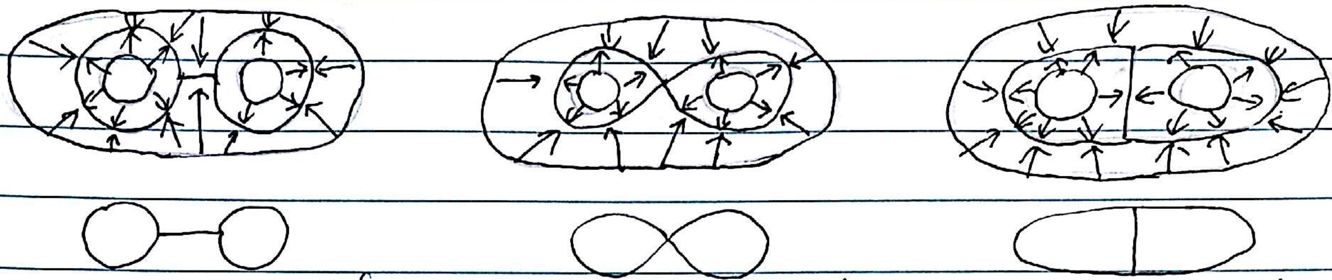
  $(x, t) \mapsto f_t(x)$

- A subspace $A \subset X$ is called a deformation retract of X if $\exists$ a deformation retraction $f_t : X \to X$ onto A.

Examples (1) The center circle of the Möbius band M is a deformation retract of M.



(2).



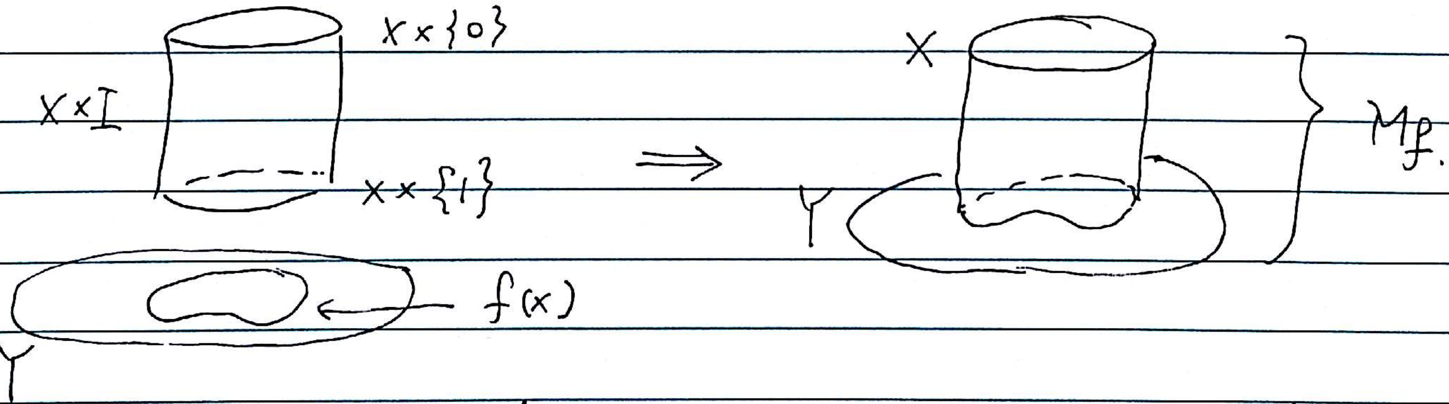
are deformation retracts of the 2 punctured disc.

Defi: Mapping cylinder M_f of a map $f: X \to Y$

$$M_f = X \times I \amalg Y \big/ \sim$$

where the identification is given by

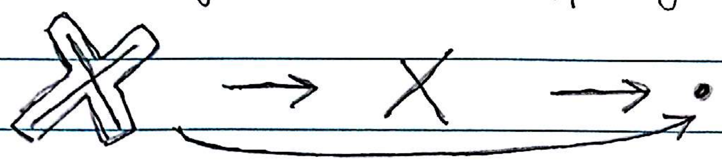
$$(x, 1) \sim f(x).$$



• Y is a deformation retract of M_f under the deformation retraction $g_t: M_f \to M_f$ defined as follows:

$$g_t: \begin{array}{l} (x, s) \longmapsto (x, s + (1-s)t) \\ y \longmapsto y \end{array}$$

- Not all deformation retractions arise in ~~this~~ way from mapping cylinders.



$\because$ certain pairs of points merge before reaching their final destination

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Def: A subspace $A \subset X$ is called a retract of X if there exists a map $r: X \to X$ such that (1) $r(x) \in A$, $x \in X$.
(2) $r(a) = a$, $a \in A$.

Such a map $r: X \to X$ is called a retraction of X onto A.

- Not all retractions comes from deformation retractions

Ex: (1) A space X always retracts onto any point x_0 via the constant map $r(x) = x_0$, $\forall x \in X$ but a space that deformation retracts onto a point must be path-connected.

(2) "Letters with holes" A, B, D, O, P, Q, R do not deformation retract onto a point.

Def: A family of maps $f_t: X \to Y$ is called a homotopy between f_0 and f_1 if the associated map

$$F: X \times I \longrightarrow Y \quad \text{s.t.} \quad F(x,t) = f_t(x)$$

is continuous.

- One says that $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy f_t connecting them.
  "$f_0 \simeq f_1$."

Def: Let $A \subset X$ a subspace. If a homotopy $f_t : X \to Y$ satisfies $f_t(a) = f_0(a)$, $\forall a \in A, t \in I$.
(i.e. $f_t|_A$ is indep. of t)
then f_t is a homotopy relative to A or simply a homotopy rel A.

- If $f_t : X \to X$ is a deformation retraction onto $A \subset X$, then $f_0 = \mathbb{1}_X : X \to X$ and f_1 is a retraction $r : X \to X$ onto A
$\Rightarrow$ A deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

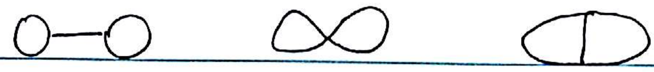
- If X deformation retracts onto $A \subset X$ via $f_t : X \to X$ then if $r : X \to A$ denotes the resulting retraction
$i : A \hookrightarrow X$ the inclusion,
$\Rightarrow$ $r \circ i = \mathbb{1}_A$, $i \circ r \simeq \mathbb{1}_X$ (the homotopy is f_t)

Def: A map $f : X \to Y$ is called a homotopy equivalence if $\exists$ a map $g : Y \to X$ s.t. $g \circ f \simeq \mathbb{1}_X$, $f \circ g \simeq \mathbb{1}_Y$.

- X and Y are said to be homotopy equivalent or have the same homotopy type.
  "$X \simeq Y$".

Check: It's an equivalence relation.

Example: (1) If A is a deformation retract of X, then $X \simeq A$.

(2).  are all homotopy equivalent. Since they are deformation retracts of the same space .

But none of the three is a def. retr. of another.

Def: (1) A map $f: X \to Y$ is said to be <u>null-homotopic</u> if it is homotopy to a constant map $c: X \to Y$.

(2) A space is called <u>contractible</u> if it has the same type of a point.

- A space is <u>contractible</u> iff its identity map is nullhomotopic.

Example: (1) D^n and $\mathbb{R}^n$ are contractible.

(2) $f, g: \mathbb{R} \to \mathbb{R}$ any two continuous maps.

$\Rightarrow f \simeq g$.

Define $F: \mathbb{R} \times I \to \mathbb{R}$ by

$$F(x,t) = (1-t) f(x) + t g(x)$$

Clearly F is continuous. ($\because f, g$ continuous)

$$F(x,0) = f(x), \quad F(x,1) = g(x).$$

$\Rightarrow F$ is a homotopy between f and g.

In particular, this shows that any continuous map $f: \mathbb{R} \to \mathbb{R}$ is null-homotopic.