Induced homomorphisms.

Suppose $\varphi: X \to Y$ is a map such that $\varphi(x_0) = y_0$.
Then φ induces a homomorphism

$$\varphi_*: \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

for any loop f in X with basepoint x_0.

$$\varphi_*[f] = [\varphi f].$$

① φ_* is well-defined

Since a homotopy f_t of loops based at x_0 yields a composed homotopy φf_t of loops based at y_0. so $\varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$.

② φ_* is a homomorphism.

Since $\varphi(f \cdot g) = (\varphi f) \cdot (\varphi g)$

both functions having values.

$$\begin{cases} \varphi f(2s), & 0 \le s \le \frac{1}{2} \\ \varphi g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

$$\varphi_*([f])\varphi_*([g]) = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*([f \cdot g])$$
$$= \varphi_*([f][g]).$$

③ φ_* is functorial.

(1) $(\varphi \circ \psi)_* = \varphi_* \psi_*$

$\because (\varphi \circ \psi)_*[f] = [\varphi \circ \psi \circ f] = \varphi_*[\psi \circ f] = \varphi_* \psi_* [f]$.

(2) $Id_* = Id$. i.e. $Id: X \to X \rightsquigarrow Id: \pi_1(X, x_0) \to \pi_1(X, x_0)$

obvious.

- If φ is a homeomorphism with inverse ψ, then φ_* is an isomorphism with inverse ψ_*.

since $\varphi_* \psi_* = (\varphi \psi)_* = Id_* = Id$.

Similarly $\psi_* \varphi_* = Id$

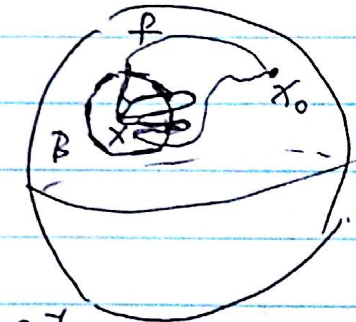
Proposition 1.14 $\pi_1(S^n) = 0$ if $n \geq 2$.

Pf: ① Let $f: I \longrightarrow S^n$ be a loop with basepoint x_0.
If $x \notin f(I)$, then f is nullhomotopic
$\because$ $S^n - \{x\}$ is homeomorphic to $\mathbb{R}^n$
which is simply-connected.

② Consider a small open ball B in S^n
about $x \neq x_0$.

$f^{-1}(B)$ is open in $(0,1)$ and is a (possibly
infinite) union of disjoint open intervals (a_i, b_i)

$f^{-1}(x)$ is compact and is contained
in the union of these intervals
$\Rightarrow$ $f^{-1}(x)$ is contained in the union
of finitely many of them.



Consider one of them (a_i, b_i) meet $f^{-1}(x)$.

$f_i = f|[a_i, b_i] \subset \overline{B}$ and $f(a_i), f(b_i) \in \partial \overline{B}$.

If $n \geq 2$, choose a path g_i from $f(a_i)$ to $f(b_i)$
in $\overline{B}$ but disjoint from x.
Since $\overline{B}$ is homeomorphic to a convex set in $\mathbb{R}^n$
hence simply-connected.
f_i is homotopic to g_i by proposition 1.6.
After repeating this process for each of (a_i, b_i)
that meet $f^{-1}(x)$ we obtain a loop g homotopic
to original f and $g(I) \cap \{x\} = \emptyset$.

#

<u>Example 1.15</u> For a point $x \in \mathbb{R}^n$.

$\mathbb{R}^n - \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$

$$y \longmapsto \left(\frac{y-x}{\|y-x\|}, \ln\|y-x\| \right)$$

By proposition 1.12, $\pi_1(\mathbb{R}^n - \{x\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$

But $\pi_1(\mathbb{R}) = 0$.

$\because$ Define $k: \mathbb{R} \times I \to \mathbb{R}$

$k_t(s) = k(s,t) = (1-t)s$

k_t : a homotopy from id to constant map at 0.

For a loop $f: I \to \mathbb{R}$ at 0

$h_t(s) = h(s,t) = k(f(s),t)$

h_t : a homotopy from f to the constant map at 0.

Hence $\pi_1(\mathbb{R}^n - \{x\}) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ 0 & \text{if } n>2 \end{cases}$

<u>Corollary 1.16</u> $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.

<u>Pf:</u>
① $n=1$. is easy.

suppose $f: \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism.

$\mathbb{R}^2 - \{0\}$ is path-connected.

but $\mathbb{R}^n - \{f(0)\}$ is not path-connected when $n=1$.

② $n \geq 2$. $\mathbb{R}^n - \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

$$\pi_1(\mathbb{R}^n - \{0\}) \cong \begin{cases} \mathbb{Z} & n=2 \\ 0 & n \geq 3. \end{cases}$$

So the spaces cannot be homeomorphic. #.

Remark: To prove that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $m = n$, we need to use homology.

proposition 1.17 ① If X retracts onto a subspace A, then the homomorphism $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i: A \hookrightarrow X$ is injective.
② If A is a deformation retract of X, then i_* is an isomorphism.

Pf: ① If $r: X \to A$ is a retraction, then $ri = Id$.
$\Rightarrow r_* i_* = Id \Rightarrow i_*$ is injective.
② If $r_t: X \to A$ is a deformation retraction of X onto A, then $r_0 = Id$, $r_t|_A = Id$, $r_1(X) \subseteq A$.
If $f: I \to X$ is any loop in X based at x_0, then $r_t \circ f$ is a homotopy of f to a loop in A, so i_* is also surjective.

Remark: we have shown earlier in the proof of the Brouwer fixed point theorem that S^1 is not a retract of D^2.
Any way to prove it:
$i: S^1 \hookrightarrow D^2$
$i_*: \pi_1(S^1) \to \pi_1(D^2)$ is a homomorphism from $\mathbb{Z} \to 0$ that cannot be injective.

<u>Proposition 1.18</u> If $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$.

<u>Lemma 1.19</u> If $\varphi_t : X \to Y$ is a homotopy and h is the path $t \to \varphi_t(x_0)$, then $(\varphi_0)_* = \beta_h (\varphi_1)_*$.

Pf: Let $h_t(s) = h(st)$ for $s \in [0,1]$ so that h_t is the restriction of h to $[0,t]$ and the reparametrizing to a path on $[0,1]$

If f is a loop in X based at x_0, $h_t \cdot (\varphi_t f) \cdot \bar{h}_t$ gives a homotopy of loops at $\varphi_0(x_0)$.

Restricting this homotopy to $t=0$ and $t=1$,

$\Rightarrow (\varphi_0)_*([f]) = \beta_h((\varphi_1)_*([f]))$. ※

<u>Pf of proposition 1.18</u>

Let $\psi : Y \to X$ be a homotopy inverse for φ, $\varphi\psi \simeq Id$, $\psi\varphi \simeq Id$.

Consider
$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

① since $\psi\varphi \simeq Id$, $\psi_* \varphi_*$ is an isomorphism.

$\Rightarrow \psi_* \varphi_* = \beta_h$ for some h, by the lemma.

Since $\psi_* \varphi_*$ is an isomorphism, φ_* is injective.

② Similarly, ψ_* is injective.

Since φ_* and ψ_* are both injective and their composition is an isomorphism. $\Rightarrow$ Both are isomorphisms.

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