Homotopy Invariance.

- For a map $f: X \to Y$, an induced homomorphism
$$f_\# : C_n(X) \to C_n(Y)$$
is defined by composing each singular n-simplex
$\sigma: \Delta^n \to X$ with f
to get a singular n-simplex $f_\#(\sigma) = f\sigma : \Delta^n \to Y$,
then extending $f_\#$ linearly via
$$f_\#\Big(\sum_i n_i \sigma_i\Big) = \sum_i n_i f_\#(\sigma_i) = \sum_i n_i f\sigma_i.$$

- The maps $f_\# : C_n(X) \to C_n(Y)$ satisfy $f_\# \partial = \partial f_\#$.

since
$$\begin{aligned} f_\# \partial(\sigma) &= f_\#\Big(\sum_i (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]\Big) \\ &= \sum_i (-1)^i f\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n] \\ &= \partial f_\#(\sigma). \end{aligned}$$

- Thus we have a diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

- The maps $f_\# : C_n(X) \to C_n(Y)$ satisfy $f_\# \partial = \partial f_\#$ means that the $f_\#$'s define a <u>chain map</u> from the singular chain complex of X to that of Y.

- $f_\# \partial = \partial f_\# \Rightarrow$ ① $f_\#$ takes cycles to cycles
$\because\ \partial\alpha = 0 \Rightarrow \partial(f_\#\alpha) = f_\#(\partial\alpha) = 0.$
② $f_\#$ takes boundaries to boundaries.

$\because\ f_\#(\partial\beta) = \partial(f_\#\beta)$.

Hence $f_\#$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$.

Prop 2.9: A chain map between chain complexes induces homomorphisms between the homology groups of two complexes.

Two basic properties of induced homomorphisms:

(i) $(fg)_* = f_* g_*$ for $X \xrightarrow{g} Y \xrightarrow{f} Z$.

This follows from associativity of $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$

(ii) $\mathbb{1}_* = \mathbb{1}$, where $\mathbb{1}$ denotes the identity map of a space or a group.

Prop 2.10: If $f, g: X \to Y$ are homotopic, then they induce the same homomorphism
$f_* = g_*: H_n(X) \to H_n(Y)$.

$(fg)_* = f_* g_*$, $\mathbb{1}_* = \mathbb{1}$.

$\Rightarrow$ Cor 2.11: $f_*: H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f: X \to Y$ are isomorphisms, $\forall n$.

For example, if X is contractible, then $\tilde{H}_n(X) = 0$, $\forall n$.

Pf of Cor 2.11

If $g: Y \to X$ is a homotopy inverse of f, then $gf \simeq id_X$ and $fg \simeq id_Y$.

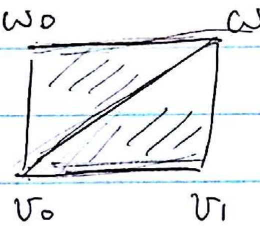
Then $g_* f_* = (id_X)_*: H_n(X) \to H_n(X)$

$f_* g_* = (id_Y)_* : H_n(Y) \to H_n(Y)$.

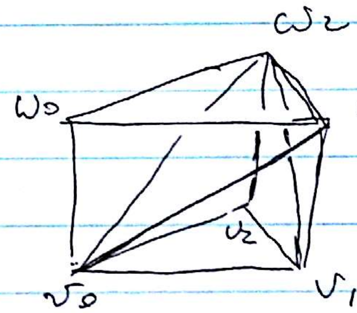
$\Rightarrow f_*$ is one-to-one and onto. #.

Pf of Prop 2.10 (Sketch)

- The product $\Delta^n \times I$ is subdivided into $(n+1)$-simplices in the following way:
  In $\Delta^n \times I$, let $\Delta^n \times \{0\} = [v_0, \cdots, v_n]$
  $\Delta^n \times \{1\} = [w_0, \cdots, w_n]$
  where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$.
- $n=1$



$n=2$



- The $(n+1)$-simplex $[v_0, \cdots, v_i, w_i \cdots w_n]$ in $\Delta^n \times I$ is the region lying between the two n-simplices $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$ and $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$.
- Given a homotopy $F: X \times I \to Y$ from f to g we define the prism operators
  $P: C_n(X) \to C_{n+1}(Y)$ by
  $$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathbb{1})\big|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]}$$
  for $\sigma: \Delta^n \to I$, where $F \circ (\sigma \times \mathbb{1})$ is the composition
  $$\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$$
- Some computation.
  $$P(\partial \sigma) + \partial(P\sigma) = g \circ \sigma - f \circ \sigma.$$

- If $\alpha \in C_n(X)$ is a cycle, then
  $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$ $\because \partial\alpha = 0$.
  $\Rightarrow g_\#(\alpha) - f_\#(\alpha)$ is a boundary.
  $\Rightarrow f_\#(\alpha)$ and $g_\#(\alpha)$ determine the same homology class.
  $\Rightarrow f_* = g_* : H_n(X) \to H_n(Y)$, $\forall n$. #

- If $\partial P + P\partial = g_\# - f_\#$, then we say that P is a <u>chain homotopy</u> between the chain maps $f_\#$ and $g_\#$.

<u>prop 2.12</u>: chain-homotopic chain maps induce the same homomorphism on homology.

<u>Remark</u>: The properties of induced homomorphisms we proved above hold equally well in the setting of reduced homology, with the same proofs.