## Short exact sequence of chain complexes

Let $A: \cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \to \cdots$

$B: \cdots \to B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \to \cdots$

$C: \cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$

be chain complexes and let $i: A \to B$ and $j: B \to C$ be chain maps such that

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$$

is exact for each $n$.

i.e.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] \arrow[dr, phantom, "\circlearrowleft"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] \arrow[dr, phantom, "\circlearrowleft"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] \arrow[dr, phantom, "\circlearrowleft"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] \arrow[dr, phantom, "\circlearrowleft"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the columns are exact, the rows are chain complexes

- Such a commutative diagram is called <u>a short exact sequence of chain complexes</u>

- We will show that this diagram will induce a long exact sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

The chain maps $i$ and $j$ induce maps $i_*$ and $j_*$ on homology.

- To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$
  connecting homomorphism
- let $c \in C_n$ be a cycle.
  $\because$ $j$ is onto, $c = j(b)$ for some $b \in B_n$

$$\begin{tikzcd}
 & & a \arrow[d, mapsto] & A_{n-1} \arrow[d, "i"] \\
 & & \partial b & \\
b \arrow[urr, mapsto, bend left] \arrow[d, mapsto] & B_n \arrow[rr, "\partial"] \arrow[d, "j"] & & B_{n-1} \\
c & \in C_n & &
\end{tikzcd}$$

- $\partial b \in B_{n-1}$ is in $\ker j$ $\because$ $j(\partial b) = \partial j(b) = \partial c = 0$.
- So $\partial b = i(a)$ for some $a \in A_n$ $\because$ $\ker j = \operatorname{Im} i$.
- Note that $\partial a = 0$ $\because$ $i(\partial a) = \partial i(a) = \partial \partial b = 0$. and $i$ is injective.

we define $\partial : H_n(C) \to H_{n-1}(A)$ by sending $[c]$ to $[a]$,

$$\partial[c] = [a].$$

This is well-defined since:

- $a$ is uniquely determined by $\partial b$ since $i$ is injective.
- A different choice $b'$ for $b \Rightarrow j(b') = j(b)$
  $\Rightarrow b' - b \in \ker j = \operatorname{Im} i \Rightarrow b' - b = i(a')$ for some $a'$.
  $\Rightarrow b' = b + i(a')$.

  Replacing $b$ by $b + i(a')$ is to change $a$ to $a + \partial a'$.
  $\because$ $i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a'))$.
- A different choice of $c$ would have the form $c + \partial c'$. $\because$ $c' = j(b')$ for some $b'$.
  $\Rightarrow c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$.
  $\partial b = \partial(b + \partial b') \Rightarrow a$ is unchanged.

The map $\partial: H_n(C) \to H_{n-1}(A)$ is a homomorphism.

$\because$ if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via $b_1$ & $b_2$.

then $j(b_1+b_2) = j(b_1) + j(b_2) = c_1 + c_2$.

$i(a_1+a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1+b_2)$

So $\partial([c_1]+[c_2]) = [a_1] + [a_2]$.

Thm 2.16 The sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

is exact.

Pf:

① $\mathrm{Im}\, i_* \subset \ker j_*$ $\quad \because ji = 0 \Rightarrow j_* i_* = 0$.

② $\mathrm{Im}\, j_* \subset \ker \partial$. We have $\partial j_* = 0$.

Suppose $b \in B_n$ is a $n$-cycle.

$\Rightarrow \partial j_*[b] = \partial[j(b)] = [a]$ for an element

$a \in A_{n-1}$ satisfying $i(a) = \partial b = 0$.

$\because i$ is inj. $\Rightarrow a = 0$.

$\partial j_*[b] = [0] = 0$.

③ $\mathrm{Im}\, \partial \subset \ker i_*$ $\quad i_* \partial[c] = i_*[a] = [i(a)] = [\partial b] = 0$.

④ $\ker j_* \subset \mathrm{Im}\, i_*$.

Let $b \in B_n$ such that $j_*[b] = 0. \Rightarrow j(b) = \partial c'$

a cycle. for some $c' \in C_{n+1}$.

$\because j$ is surjective. $c' = j(b')$ for some $b' \in B_{n+1}$.

We have $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$

$\because \partial j(b') = \partial c' = jb$.

So $b - \partial b' = i(a)$ for some $a \in A_n$.

$a$ is a cycle $\because i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$

and $i$ is injective.

$i_*[a] = [b - \partial b'] = [b]$. $\Rightarrow$ $i_*$ maps onto $\ker j_*$.

⑤ $\ker \partial \subset \operatorname{Im} j_*$.

Suppose $\partial[c]$ for some $n$-cycle $c \in C_n$.
Then the $(n-1)$-cycle $a \in A_{n-1}$ representing $\partial[c]$ is an $(n-1)$-boundary. $a = \partial a'$ for some $a' \in A_n$.

If $b \in B_n$ satisfies $jb = c$ and $ia = \partial b$,
then $\partial(b - i(a')) = \partial(b) - i\partial(a') = \partial(b) - i(a) = 0$
and $j(b - ia') = jb = c$.
Hence $[c] = [j(b - i(a'))] = j_*[b - i(a')]$.
$\Rightarrow \ker \partial \subset \operatorname{Im} j_*$.

⑥ $\ker i_* \subset \operatorname{Im} \partial$.

Let $i_*[a] = 0$ for an $(n-1)$-cycle $a \in A_{n-1}$.
Then $i(a) = \partial b$ for some $b \in B_{n-1}$.
$\partial j(b) = j\partial b = jia = 0$.
$\Rightarrow j(b)$ is a cycle in $C_n$.
$\partial$ takes $[j(b)]$ to $[a]$. #.

- This theorem represents the beginnings of the subject of homological algebra.
- The method of proof is sometimes called <u>diagram chasing</u>