Mayer–Vietoris sequences

For a pair of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B, there is an exact sequence

$$0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A)\oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

with φ and ψ defined by

$$\varphi(x) = (x, -x). \quad \text{and} \quad \psi(x, y) = x+y.$$

since $\varphi\partial(x) = (\partial x, -\partial x) = (\partial\oplus\partial)\varphi(x)$.

$\partial : C_n(x) \to C_{n-1}(x)$. takes $C_n(A\cap B) \to C_{n-1}(A\cap B)$

$C_n(A+B) \to C_{n-1}(A+B)$.

$$\partial\psi(x, y) = \partial x + \partial y = \psi(\partial\oplus\partial)(x, y).$$

We obtain the long exact sequence.

$$\to H_{n+1}(X) \xrightarrow{\partial} H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(x) \xrightarrow{\partial} \cdots$$

$$\cdots \to H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(X) \to 0.$$

It is called the Mayer–Vietoris sequence.

The homomorphisms are defined as follows:

$$\Phi(a) = (a, -a), \quad \Psi(a, b) = a+b.$$

For $z \in H_n(x)$, $z = [x+y]$ for some $x \in C_n(A)$

$y \in C_n(B)$

satisfying $\partial x = -\partial y$.

Define $\partial z = [\partial x]$.

We also have the Mayer–Vietoris sequence of reduced homology groups.

The Mayer–Vietoris sequence is valid when A, B are A, B are closed subsets of X satisfying

(1) $A \cup B = X$

(2) A and B are deformation retracts of neighborhoods U and V with $U \cap V$ deformation retracting onto $A \cap B$.

Example 2.46: Take $X = S^n$, A, B the northern and southern hemispheres so that $A \cap B = S^{n-1}$.

The reduced Mayer–Vietoris sequence

$$\to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(S^n) \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B)$$

reduces to $0 \to \tilde{H}_n(S^n) \to \tilde{H}_{n-1}(S^{n-1}) \to 0$.

Therefore $\tilde{H}_n(S^n) \xrightarrow{\simeq} \tilde{H}_{n-1}(S^{n-1}) \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \tilde{H}_0(S^0) \approx \mathbb{Z}$.

Example 2.47

The Klein bottle K is the union of two Möbius bands A, B with $A \cap B = \partial A = \partial B$ (boundary circle).

The reduced Mayer–Vietoris sequence is

$$H_2(A) \oplus H_2(B) \to H_2(K) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(K) \to 0$$

Since A, B and $A \cap B$ are all homotopy equivalent to circles.

The exact sequence reduces to

$$0 \to H_2(K) \to \mathbb{Z} \xrightarrow{\Phi} \mathbb{Z} \oplus \mathbb{Z} \to H_1(K) = 0$$

$\Phi(1) = (2, -2)$ $\because$ The boundary circle of a Möbius band wraps around the core circle twice.

Since Φ is injective, $H_2(K) = 0$. Using the basis $\{(1,0), (1,-1)\}$ for $\mathbb{Z} \oplus \mathbb{Z}$, we get $H_1(K) \approx \mathbb{Z} \oplus \mathbb{Z}_2$.

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