Review on cell (CW) complex.

e^n: n-cell. e^n is homeomorphic to $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$

$D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$.

$\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.

$D^0 = e^0$. $\bullet$ $\qquad D^1 = e^0 \overset{e^1}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} e_0 = e^1 \cup S^0$.

$\partial D^1 = S^0$, $\underset{e_0}{\bullet} \quad \underset{e_0}{\bullet}$.

Cell complex.

(1) start with a set of points X^0.

(2) Form X^1 by attaching 1-cells to X^0.

Form X^2 by attaching 2-cells to X^1.

$\vdots$

Form X^n by attaching n-cells to X^{n-1}

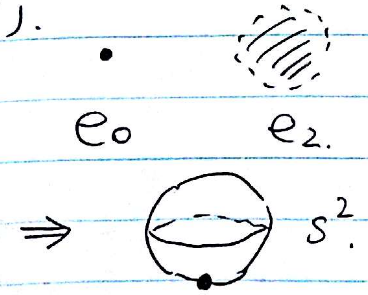
$\vdots$

- Attaching n-cells to X^{n-1} via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$

$$\underset{\substack{\uparrow \\ n\text{-skeleton}}}{X^n} = \frac{X^{n-1} \amalg_\alpha D^n_\alpha}{x \sim \varphi_\alpha(x)} \;\underset{\substack{\uparrow \\ \text{as a set}}}{=\!=}\; X^{n-1} \amalg_\alpha e^n_\alpha.$$

Ex: S^2.

(1).



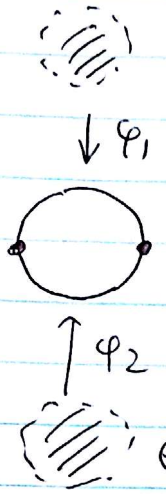
$X^0 = X^1$. $\qquad \varphi : S^1 \to X^1 = X^0 = e_0$.

$X^2 = \dfrac{X^1 \amalg D^2}{x \sim \varphi(x)} = S^2 = e^0 \cup e^2$.

$x \in S^1 = \partial D^2$.

$S^n = e^0 \cup e^n$.

(2).



e^2

$\downarrow \varphi_1$

$S^1 = e^0 \cup e^0 \cup e^1 \cup e^1 \Rightarrow$

S^2.

$\uparrow \varphi_2$

e_2

$$X^2 = \frac{X^1 \amalg D_\alpha^2}{x \sim \varphi_\alpha(x)}, \quad \alpha = \{1, 2\}, \quad x \in \partial D_\alpha^2 = S_\alpha^1.$$

$$S^2 = e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^2 \cup e^2.$$

$$S^n = e^0 \cup e^0 \cup \cdots \cup e^n \cup e^n.$$

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$

$\Phi_\alpha : D_\alpha^n \longrightarrow X$: characteristic map extending the attaching map $\varphi_\alpha : \underset{\displaystyle S_\alpha^{n-1}}{\underset{\shortparallel}{\partial D_\alpha^n}} \to X$

$\Phi_\alpha \big|_{\mathrm{int}(D_\alpha^n)}$ is homeomorphic to e_α^n.

$\mathrm{int}(D_\alpha^n)$: interior of D_α^n

$$\begin{tikzcd} D_\alpha^n \arrow[r, hook] \arrow[rrr, bend right, "\Phi_\alpha"'] & X^{n-1} \amalg_\alpha D_\alpha^n \arrow[r, "\text{quotient map}"] & X^n = \dfrac{X^{n-1} \amalg_\alpha D_\alpha^n}{x \sim \varphi_\alpha(x)} \arrow[r, hook] & X \end{tikzcd}$$

- CW pair (X, A): X : a cell complex.
  A : a subcomplex of X.

Ex: $\mathbb{RP}^i \subset \mathbb{RP}^n$, $i \le n$. $(\mathbb{RP}^n, \mathbb{RP}^i)$ is a CW pair.

- A subcomplex A of X is a closed subspace of X that is a union of cells of X.

Let e^n be a cell in A.

Since $\Phi|_{\mathrm{int}(D^n)} \underset{\text{homeom.}}{\cong} e^n$ and A is closed.

$$\Phi|_{\partial D^n} \subset A.$$

$\Rightarrow$ A is a cell complex in its own.

Operations on spaces

Products X, Y: cell complexes.

Then $X \times Y$ has the structure of a cell complex with cells $e^m_\alpha \times e^n_\beta$, where e^m_α ranges over cells of X

e^n_β " " " " Y.

Ex: if $S^1 = e^0 \cup e^1$ 

then $S^1 \times S^1 = \underset{\text{0-cell}}{(e^0 \times e^0)} \cup \underset{\text{1-cell}}{(e^1 \times e^0)} \cup \underset{\text{1-cell}}{(e^0 \times e^1)} \cup \underset{\text{2 cell}}{(e^1 \times e^1)}$

$= \text{torus } T^2.$

Quotient. if (X, A) is a CW-pair.

then X/A inherits a natural cell complex structure from X.

The cells of X/A = The cells of $X - A$
+ 1 new 0-cell
(the image of A in X/A).

Ex: $D^n / S^{n-1} = S^n$.

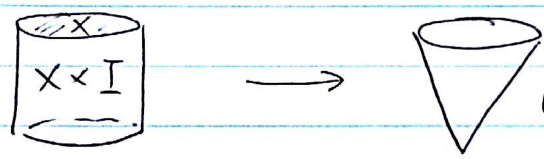
D_2  s^1. (D^2, S^1) a CW pair. $\quad D^2/S^1 = S^2$.

Cone & suspension.

given a space X, the cone on X, denoted by CX, is the quotient space

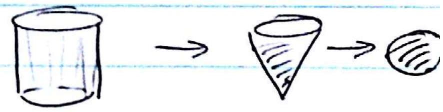
$$CX = X \times I / X \times \{0\}.$$

$X \times \{1\}$



CX.

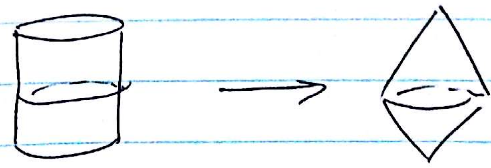
$X \times \{0\}$

Ex: $CS^{n-1} = D^n$. $\quad CS^1 = D^2$ 

given a space X, the suspension of X, denoted by SX, is the quotient space

$$SX = X \times I \Big/ \begin{matrix} (x,0) \sim (y,0) \\ (x,1) \sim (y,1) \end{matrix} \qquad x, y \in X.$$

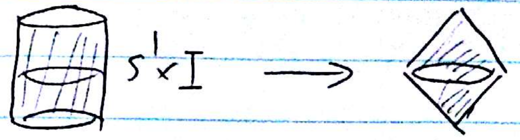
$X \times \{1\}$



$X \times \{0\}$

suspension is the union of two cones.

Ex: $S S^n = S^{n+1}$.



S^2.

<u>Wedge sum</u>:

If $x_0 \in X$, $y_0 \in Y$ the wedge sum $X \vee Y$ is the quotient space $X \amalg Y / x_0 \sim y_0$

<u>EX</u>: $S^1 \vee S^1$

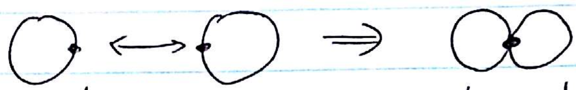
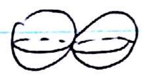
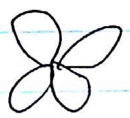


figure 8.

S^1 S^1 $S^1 \vee S^1$

$S^1 \vee S^2$  ,  $S^2 \vee S^2$.

<u>Ex</u>: wedge of n circles.

 $n=4$.

<u>Smash product</u> $x_0 \in X$, $y_0 \in Y$.

The smash product of X and Y, denoted as $X \wedge Y$, is the quotient space $X \wedge Y = \dfrac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}$

$$= \frac{X \times Y}{X \vee Y}.$$

intersect at (x_0, y_0).

<u>Ex</u>: $S^m \wedge S^n = \dfrac{S^m \times S^n}{S^m \vee S^n} = S^{m+n}$.

$$\frac{S^m \times S^n}{S^m \vee S^n} = \frac{(e^0 \times e^0) \cup (e^m \cancel{\times e^0}) \cup \cancel{(e^0 \times e^n)} \cup (e^m \times e^n)}{\cancel{(e^0 \times e^0)} \cup (e^m \cancel{\times e^0}) \cup \cancel{(e^0 \times e^m)}}$$

$$= e^0 \cup e^{m+n}.$$

$\dfrac{S^1 \times S^1}{S^1 \vee S^1} = S^2.$