Chapter 1. The fundamental group

§1.1 Basic constructions

Paths and homotopy

Def: • A path in a space X is a map $f: I \to X$, where $I = [0,1]$.

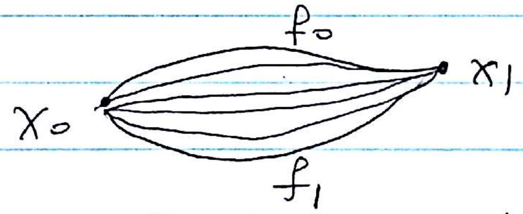
• A homotopy of paths in X is a family $f_t: I \to X$, $0 \le t \le 1$. such that

(1) The endpoints $f_t(0) = x_0$, $f_t(1) = x_1$ are independent of t.

(2). The associated map $F: I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.

• f_t is called a homotopy between f_0 and f_1.

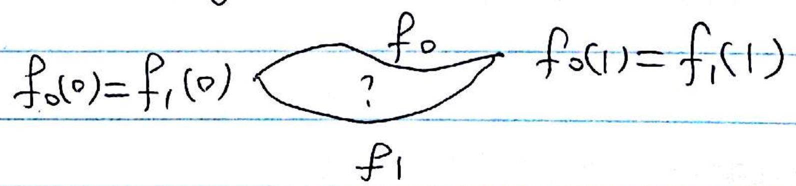
Notation: $f_0 \simeq f_1$



Example 1.1: Linear homotopy

Suppose f_0, f_1 are paths in $\mathbb{R}^n$ such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(0)$.

Then f_0 and f_1 are homotopic via the homotopy $f_t(s) = (1-t) f_0(s) + t f_1(s) \in \mathbb{R}^n$.



Recall: A subspace $X \subseteq \mathbb{R}^n$ is called convex if $(1-t)x_0 + tx_1 \in X$ whenever $x_0, x_1 \in X$ and $0 \le t \le 1$.

If $X \subseteq \mathbb{R}^n$ is convex and if $f_0, f_1 : I \to X$ are two paths with $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$, then they are homotopic via the linear homotopy

$$f_t(s) = (1-t) f_0(s) + t f_1(s).$$

<u>prop 1.2</u>: The relation of homotopy on paths with fixed points in any space is an equivalence relation.

<u>Pf</u>: (1) <u>Reflexivity</u>: $f \simeq f$ by the constant homotopy $f_t \simeq f$.

(2). <u>symmetry</u>: $f_0 \simeq f_1 \Rightarrow f_1 \simeq f_0$

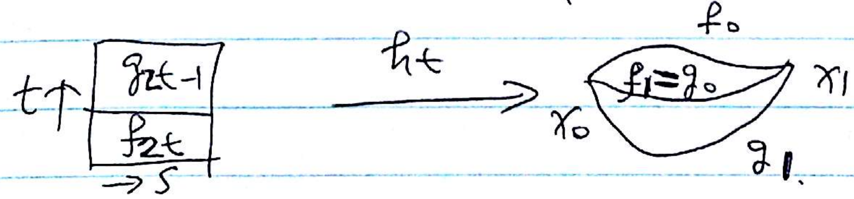
if $f_t : I \to X$ is a homotopy between f_0 and f_1

then $f_{1-t} : I \to X$ 〃 〃 〃 f_1 and f_0.

(3). <u>Transitivity</u>: $f_0 \simeq f_1$ via f_t and $f_1 = g_0 \simeq g_1$ via g_t, then $f_0 \simeq g_1$ via the homotopy h_t such that
$$h_t(s) = \begin{cases} f_{2t}(s), & 0 \le t \le \frac{1}{2} \\ g_{2t-1}(s), & \frac{1}{2} \le t \le 1. \end{cases}$$

Since $t = \frac{1}{2}$, $h_{\frac{1}{2}} = f_1 = g_0$.

The associated map $F(s,t) = h_t(s)$ is continuous.



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The equivalence class of a path f is called a <u>homotopy class of f</u> & is denoted by $[f]$.

<u>product path</u>:

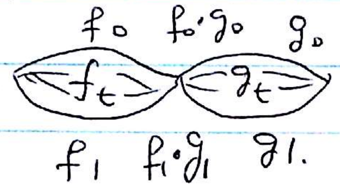
suppose $f, g : I \to X$ are paths such that $f(1) = g(0)$

Then we can define their product $f \cdot g$ by

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $f \cdot g(\frac{1}{2}) = f(1) = g(0)$,
$f \cdot g$ is well-defined and continuous.

- If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ in X with $f_0(1) = g_0(0)$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.



Pf: Exercise. #

- Therefore, if f, g are paths in X such that $f(1) = g(0)$, then $[f][g] := [f \cdot g]$.
- Let $x_0 \in X$ and let $f: I \to X$ be a path such that $f(0) = f(1) = x_0$, we call such a path a <u>loop with base point x_0</u>.

- The set of homotopy classes $[f]$ of loops $f: I \to X$ at the basepoint x_0, denoted $\pi_1(X, x_0)$, is called <u>the fundamental group of X at the basepoint x_0</u>.

<u>Remark</u>: $\pi_1(X, x_0)$ is the first in a sequence of groups $\pi_n(X, x_0)$, called <u>homotopy groups</u> (chapter 4).

(I is replaced by I^n.)

<u>Proposition 1.3</u> $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.

<u>Pf:</u> ① <u>closedness</u>

Obviously, the product of two loops with the same basepoint is also a loop with the same basepoint

Since $[f \cdot g]$ depends on $[f]$ and $[g]$ only, so $[f][g] = [f \cdot g]$ is well-defined.

② <u>Associativity</u>

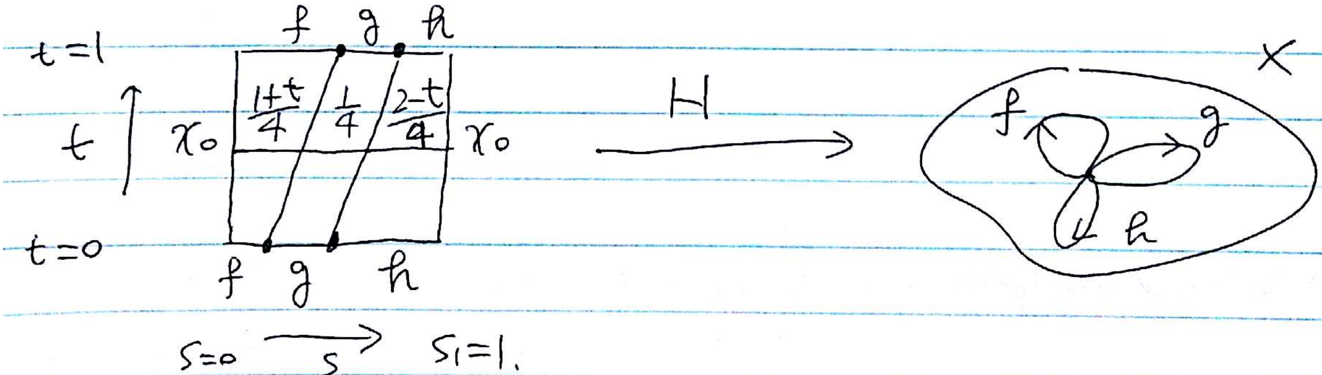
$([f][g])[h] = [f]([g][h])$ if $[f], [g], [h] \in \pi_1(X, x_0)$

To show this, it is enough to construct a homotopy such that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$.

<u>Note:</u> product of paths is <u>not</u> associative.

i.e. $(f \cdot g) \cdot h \neq f \cdot (g \cdot h)$.

But $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are homotopic.



$$H(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right), & 0 \le s \le \frac{1+t}{4} \\ g(4s-1-t), & \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ h\left(\frac{4s-2-t}{2-t}\right), & \frac{2+t}{4} \le s \le 1. \end{cases}$$

continuous.

check: $H(s,0) = (f\cdot g)\cdot h(s) = \begin{cases} f(4s), & 0 \le s \le \frac{1}{4} \\ g(4s-1), & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$

$H(s,1) = f\cdot(g\cdot h)(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le 3/4 \\ h(4s-3), & \frac{3}{4} \le s \le 1. \end{cases}$

$H(0,t) = f(0) = x_0.$

$H(1,t) = h(1) = x_0.$

One checks that $(f\cdot g)\cdot h \simeq f\cdot(g\cdot h)$ via H

$\therefore ([f][g])[h] = [f]([g][h]).$

③ The identity element.

Let $c: I \longrightarrow X$ be the constant path at x_0, denoted by $c(s) = x_0$, $s \in I$.

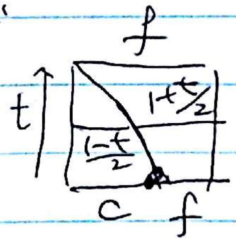
Then $[c]$ is the identity element of $\pi_1(X, x_0)$:

(1) $[c][f] = [f]$, $[f] \in \pi_1(X, x_0)$

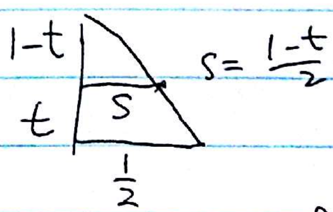
(2) $[f][c] = [f]$.

check:

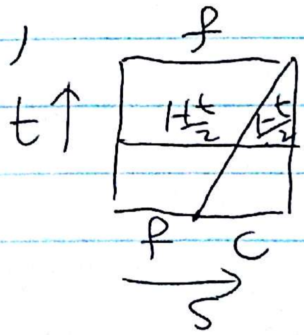
(1)



$(s,t) \longmapsto \begin{cases} x_0. & 0 \le s \le \frac{1-t}{2} \\ f\left(\dfrac{s-\frac{1-t}{2}}{\frac{1+t}{2}}\right), & \frac{1-t}{2} \le s \le 1. \end{cases}$



(2)



$(s,t) \longmapsto \begin{cases} f\left(\dfrac{s}{\frac{1+t}{2}}\right), & 0 \le s \le \frac{1+t}{2} \\ x_0, & \frac{1+t}{2} \le s \le 1. \end{cases}$

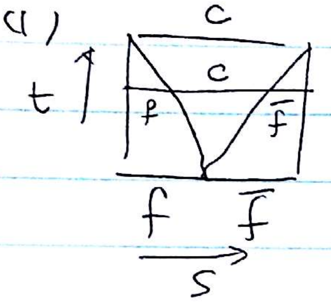
④ Existence of inverse.

Let $\overline{f}(s) = f(1-s)$ for any loop f with basepoint x_0.
Then $[f]^{-1} = [\overline{f}]$.

Check: (1) $[f][\overline{f}] = [c]$

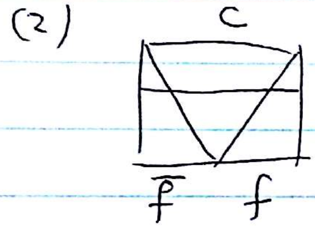
(2) $[\overline{f}][f] = [c]$.

(1)



$$(s,t) \longmapsto \begin{cases} f(2s), & 0 \le s \le \frac{1-t}{2} \\ f(1-t), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ f(2-2s), & \frac{1+t}{2} \le s \le 1. \end{cases}$$

(2)



$$(s,t) \longmapsto \begin{cases} f(1-2s), & 0 \le s \le \frac{1-t}{2} \\ f(t), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ f(2s-1), & \frac{1+t}{2} \le s \le 1. \end{cases}$$

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Example 1.4.

For any convex set X in $\mathbb{R}^n$ with basepoint x_0.
we have $\pi_1(X, x_0) = 0$.
For any loop f in X with basepoint x_0,
consider the linear homotopy

$$f_t(s) = (1-t)f(s) + t x_0.$$

Therefore, $f \simeq c$.

$\Longrightarrow \pi_1(X, x_0) = \{[c]\} = 0$.

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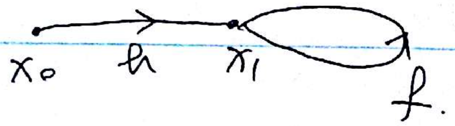
Change of basepoint

Let $h : I \to X$ be a path from x_0 to x_1
and let $f : I \to X$ be a loop with basepoint x_1



then the path $h \cdot f \cdot \bar{h} : I \to X$ defined by

$$h \cdot f \cdot \bar{h}(s) = \begin{cases} h(3s), & 0 \le s \le \frac{1}{3} \\ f(3s-1), & \frac{1}{3} \le s \le \frac{2}{3} \\ \bar{h}(3-3s), & 2/3 \le s \le 1. \end{cases}$$

is a loop with basepoint x_0.

- Note that $h \cdot f \cdot \bar{h}$ is homotopic to $(h \cdot f) \cdot \bar{h}$ and $h \cdot (f \cdot \bar{h})$.

<u>Proposition 1.5</u> Let h be a path in X from x_0 to x_1. The change-of-basepoint map $\beta_h([f]) = [h \cdot f \cdot \bar{h}]$ is an isomorphism.

<u>Pf:</u> ① <u>β_h is a homomorphism.</u>

Since $\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [\underline{h \cdot f \cdot \bar{h}} \cdot \underline{h \cdot g \cdot \bar{h}}]$
$= \beta_h[f]\, \beta_h[g]$.

② <u>β_h is an isomorphism with inverse $\beta_{\bar{h}}$.</u>

Since $\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}] = [f]$

Similarly, $\beta_{\bar{h}} \beta_h[f] = [f]$. #

- If X is path-connected, then $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of basepoint x_0.

In this case, $\pi_1(X, x_0)$ is abbreviated to $\pi_1(X)$.

(X is called <u>path-connected</u> if there exists a path in X joining any given pair of points of X)

- A space X is called <u>simply-connected</u> if it is ① path-connected and ② $\pi_1(X)=0$.

<u>Proposition 1.6</u>. A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.

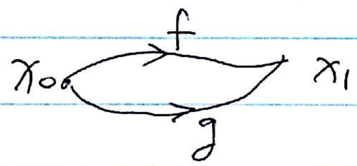
Pf: $\Rightarrow$]

Suppose X is simply-connected.

If f and g are two paths from x_0 to x_1,

then $f \simeq f \cdot \bar{g} \cdot g$ since $\bar{g}\cdot g \simeq x_1$

$\simeq g$ since $[f\cdot\bar{g}] \in \pi_1(X, x_0) = 0$.



$\Leftarrow$]. Suppose there is only one homotopy class of paths joining given pair of points of X.

Then there is only one homotopy class of loops with basepoint at x_0.

Therefore, $\pi_1(X, x_0)=0$.

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