Relative Homology groups

Given a space X and a subspace $A \subset X$.
Let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$.
Thus chains in A are trivial in $C_n(X, A)$.

Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$
takes $C_n(A)$ to $C_{n-1}(A)$,
$\Rightarrow$ induces a quotient boundary map
$$\partial : C_n(X, A) \to C_{n-1}(X, A).$$
We have a sequence of boundary maps
$$\dashrightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$
So we have a chain complex, the homology groups $H_n(X, A)$ of ~~this~~ chain complex are called <u>the relative homology groups</u> of the pair (X, A).

- Elements of $H_n(X, A)$ are represented by <u>relative cycles</u>: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial in $H_n(X, A)$.
$\Longleftrightarrow$ It is <u>a relative boundary</u>: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

$H_n(X, A)$ is "homology of X modulo A."

The inclusion map $A \to X$ induces injective homomorphisms $C_n(A) \xrightarrow{i} C_n(X)$

and the quotient homomorphisms

$$C_n(X) \xrightarrow{j} C_n(X)/C_n(A) = C_n(X, A)$$

They form an exact sequence of chain maps

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X, A) & \to & 0 \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
0 & \to & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \to & 0. \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_0(A) & \longrightarrow & C_0(X) & \longrightarrow & C_0(X, A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Theorem 2.16, we have an exact sequence.

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to$$

$$\xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X)$$

$$\xrightarrow{j_*} H_0(X, A) \to 0$$

The boundary map $\partial : H_n(X, A) \to H_{n-1}(A)$ has a very simple description:

If $[\alpha] \in H_n(X, A)$ is represented by a relative cycle α, $\because \partial\alpha \in C_{n-1}(A)$. $\partial\alpha$ is a cycle in A.

$$\Rightarrow \partial[\alpha] = [\partial\alpha].$$

$$\begin{tikzcd}
 & \alpha \arrow[r, mapsto] & {[\alpha + C_n(A)] \in H_n(X,A)} & \\
 & C_n(X) \arrow[r] \arrow[d, "\partial"] & C_n(X)/C_n(A) \arrow[r] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) &
\end{tikzcd}$$

$$\partial\alpha \longmapsto \partial\alpha$$

Long exact sequence of <u>reduced</u> homology groups:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_0(A) \arrow[r] \arrow[d] & C_0(X) \arrow[r] \arrow[d] & C_0(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

$$\Rightarrow \quad \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

$$\cdots \to H_1(X,A) \xrightarrow{\partial} \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{j_*} H_0(X,A) \to 0.$$

<u>Example 2.17</u>

The long exact sequence of reduced homology groups for $(D^n, \partial D^n)$ contains

$$\cdots \to \tilde{H}_i(D^n) \to H_i(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{i-1}(\underset{\partial D^n}{\underset{\|}{S^{n-1}}}) \to \tilde{H}_{i-1}(D^n) \to \cdots$$

Since $\tilde{H}_i(D^n) = 0$ for all i.

∂ is an isomorphism.

$$H_i(D^n, \partial D^n) \approx \begin{cases} \mathbb{Z}, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.18

The long exact sequence of reduced homology groups for (X, x_0), $x_0 \in X$, contains

$$\longrightarrow \tilde{H}_n(x_0) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} H_n(X, x_0) \to \tilde{H}_{n-1}(x_0) \to \cdots$$

Since $\tilde{H}_i(x_0) = 0$ for all i.

We have the isomorphism $j_* : \tilde{H}_n(X) \xrightarrow{\cong} H_n(X, x_0)$.

Induced homomorphisms for relative homology

- A map $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$.
- f induces homomorphisms $f_\# : C_n(X, A) \to C_n(Y, B)$ since the chain map $f_\# : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$, so we get a well-defined map on quotients, $f_\# : C_n(X, A) \to C_n(Y, B)$.
- If α is a relative cycle, i.e. $\alpha \in C_n(X)$, $\partial\alpha \in C_{n-1}(A)$, then $\partial f_\# \alpha = f_\# \partial \alpha \in f_\# C_{n-1}(A) \subseteq C_{n-1}(B)$

  i.e. $f_\#\alpha$ is a relative cycle on (Y, B).
- If α is a relative boundary, i.e. $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$, $\gamma \in C_n(A)$.

  $$f_\#\alpha = f_\#\partial\beta + f_\#\gamma = \partial f_\#\beta + f_\#\gamma.$$

  with $f_\#\beta \in C_{n+1}(Y)$ and $f_\#\gamma \in C_n(B)$.

  i.e. $f_\#\alpha$ is a relative boundary on (Y, B).
- $f_\#$ induces the homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$.

Prop 2.19. If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.

pf:

The prism operator P from the proof of theorem 2.10 takes $C_n(A)$ to $C_{n+1}(B)$. hence induces a relative prism operator

$P : C_n(X, A) \to C_{n+1}(Y, B)$.

Recall:

Given a homotopy $F : X \times I \to Y$ from f to g.

$P : C_n(X) \to C_{n+1}(Y)$ is defined by

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}) \big| [v_0, \dots, v_i, w_i, \dots, w_n]$$

for $\sigma : \Delta^n \to X$, $\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$

$\Rightarrow \partial P + P \partial = g_\# - f_\#$.

P is a chain homotopy between $f_\#$ and $g_\#$.

Since we are just passing to quotient groups, the formula $\partial P + P\partial = g_\# - f_\#$ remains valid.

Thus the maps $f_\#$ and $g_\#$ on relative chain groups are chain homotopic, and hence they induce the same homomorphism on relative homology groups.

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<u>Long exact sequence of the triple (X, A, B)</u>

If $B \subset A \subset X$, then we have the short exact sequences $0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$.
which yield the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots$$

For example, taking B to be a point,
the long exact sequence of the triple (X, A, B)
becomes the long exact sequence of reduced homology for (X, A).