Simplicial Homology

Goal: To define the simplicial homology of a Δ-complex X.

Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e_α^n of X.

Elements of $\Delta_n(X)$, called n-chains, can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$, $n_\alpha \in \mathbb{Z}$.

Equivalently, we could write $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e_α^n, with image the closure of e_α^n

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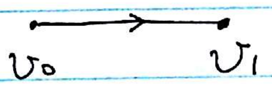
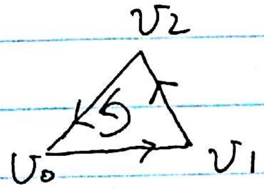
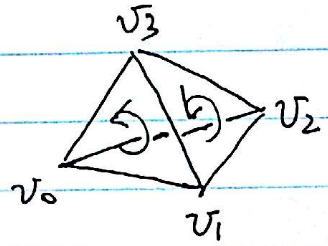
For an n-simplex $[v_0, v_1, \cdots, v_n]$, the boundary of $[v_0, v_1, \cdots, v_n]$, denoted by $\partial_n[v_0, \cdots, v_n]$, is given by $\partial_n[v_0, \cdots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i \cdots v_n]$.

Example: 

$\partial[v_0, v_1] = [v_1] - [v_0]$



$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

We define for a general Δ-simplex X a boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

by $\partial_n(\sigma_n) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \hat{v}_i, \dots, v_n]$.

Note that $\sigma_\alpha | [v_0, \dots, \hat{v}_i, \dots, v_n]$ is the characteristic map of an $(n-1)$-simplex of X.

<u>Lemma 2.1</u>: The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero. " $\partial_{n-1} \circ \partial_n = 0$ "

<u>Pf</u>: $\partial_{n-1}\partial_n(\sigma) = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]\right)$

$$= \sum_{j<i} (-1)^i (-1)^j \sigma | [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma | [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n] = 0. \quad \#$$

A <u>chain complex</u> $\mathcal{C}$ is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n.

$\partial_n \partial_{n+1} = 0 \Rightarrow \operatorname{Im} \partial_{n+1} \subset \ker \partial_n$.

We define the <u>nth homology group</u> of $\mathcal{C}$ to be the quotient group $H_n(\mathcal{C}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$.

Elements of $\ker \partial_n$ are called <u>cycles</u>.

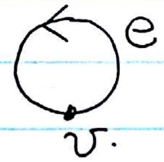
Elements of $\operatorname{Im} \partial_{n+1}$ are called <u>boundaries</u>.

Elements of $H_n(\mathcal{C})$ are called <u>homology classes</u>.

Two cycles representing the same homology class are said to be <u>homologous.</u>
This means their difference is a boundary.

Return to the case that $C_n = \Delta_n(X)$.
the homology group $H_n^{\Delta}(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$
is called the <u>nth simplicial homology group of X.</u>

<u>Example 2.2</u>: $X = S^1$.



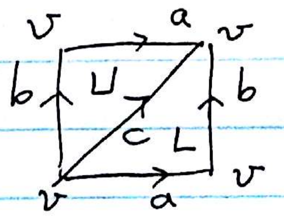
S^1 has 1 0-simplex v and 1 1-simplex e

$\partial_1 e = v - v = 0$.

$$\to \Delta_3(S^1) \xrightarrow{\partial_3} \Delta_2(S^1) \xrightarrow{\partial_2} \Delta_1(S^1) \xrightarrow{\partial_1} \Delta_0(S^1) \xrightarrow{\partial_0} 0$$
$$\Delta_3(S^1) \cong 0, \quad \Delta_2(S^1) \cong 0, \quad \Delta_1(S^1) \cong \mathbb{Z}, \quad \Delta_0(S^1) \cong \mathbb{Z}$$

$$H_n^{\Delta}(S^1) \cong \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & n \geq 2 \end{cases}$$

<u>Example 2.3</u>: $X = T$, the torus. $S^1 \times S^1$.



one vertex v
three edges a, b, c.
two 2-simplices U, L.

$$\to \Delta_3(T) \xrightarrow{\partial_3} \Delta_2(T) \xrightarrow{\partial_2} \Delta_1(T) \xrightarrow{\partial_1} \Delta_0(T) \xrightarrow{\partial_0} 0$$
$$\Delta_3(T) = 0, \quad \Delta_2(T) = \operatorname{span}\{U, L\}, \quad \Delta_1(T) = \operatorname{span}\{a, b, c\}, \quad \Delta_0(T) = \operatorname{span}\{v\}.$$

$$\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0.$$

the 1-cycles $\ker \partial_1 = \text{span}\{a, b, c\} = \Delta_1(T)$.

$$\partial_2(U) = a + b - c = \partial_2(L).$$

$$H_0^{\Delta}(T) = \Delta_0(T) \big/ \text{Im}\,\partial_1 \simeq \mathbb{Z}.$$

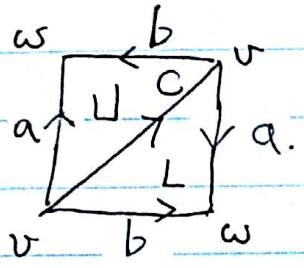
$$H_1^{\Delta}(T) = \text{span}\{a, b, c\} \big/ \text{span}\{a+b-c\}$$
$$\simeq \mathbb{Z} \oplus \mathbb{Z}.$$

$$\partial_2(pU + qL) = (p+q)(a+b-c) = 0 \text{ only if } p = -q.$$
$$\therefore \ker \partial_2 = \text{span}\{U - L\}.$$

$$H_2^{\Delta}(T) = \text{span}\{U, L\} \big/ \text{span}\{U-L\} \cong \mathbb{Z}.$$

Thus $H_n^{\Delta}(T) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & n = 1. \\ 0, & \text{otherwise.} \end{cases}$

Example 2.4 : $X = \mathbb{R}P^2$, the projective plane.



2 0-simplices v, w

3 1-simplices a, b, c

2 2-simplices U, L.

$$\text{Im}\,\partial_1 = \text{span}\{w - v\}.$$

$$\to \underset{\substack{\| \\ 0}}{\Delta_3(\mathbb{R}P^2)} \xrightarrow{\partial_3} \underset{\substack{\wr\wr \\ \text{span}\{U, L\}}}{\Delta_2(\mathbb{R}P^2)} \xrightarrow{\partial_2} \underset{\substack{\wr\wr \\ \text{span}\{a, b, c\}}}{\Delta_1(\mathbb{R}P^2)} \xrightarrow{\partial_1} \underset{\substack{\wr\wr \\ \text{span}\{v, w\}}}{\Delta_0(\mathbb{R}P^2)} \xrightarrow{\partial_0} 0$$

$H_0^{\Delta}(X) \simeq \mathbb{Z}$.

$\ker \partial_1 = \text{span}\{a-b, c\} \simeq \mathbb{Z} \oplus \mathbb{Z}$.

$\partial_2 U = -a + b + c, \qquad \partial_2 L = a - b + c$.

∂_2 is injective.

$H_2^{\Delta}(X) = 0$.

We can choose c and $a-b+c$ as a basis for $\ker \partial_1$. and $a-b+c$ and $2c = (a-b+c) + (-a+b+c)$ as a basis for $\text{Im}\, \partial_2$.

$$H_1^{\Delta}(X) = \ker\partial_1 / \text{Im}\,\partial_2 = \text{span}\{a-b+c, c\} / \text{span}\{a-b+c, 2c\}$$

$$\simeq \mathbb{Z}_2.$$

$$H_n^{\Delta}(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}_2, & n=1. \\ 0, & \text{otherwise.} \end{cases}$$

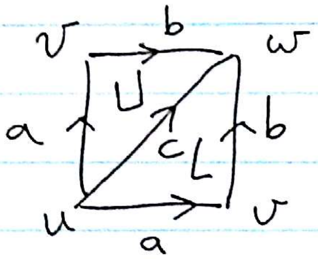
<u>Example 2.5</u>: $X = S^n$.

We obtain a Δ-complex structure on S^n by taking <u>two copies of Δ^n</u> and identifying their boundaries via the identity map.

Labeling these 2 n-simplices U, L.

$\ker \partial_n = \text{span}\{U - L\}$.

<u>Consider $n=2$</u>.



3 0-simplices. u, v, w

3 1-simplices. a, b, c

2 2-simplices. U, L.

$$\to \Delta_3(S^2) \xrightarrow{\partial_3 = 0} \Delta_2(S^2) \xrightarrow{\partial_2} \Delta_1(S^2) \xrightarrow{\partial_1} \Delta_0(S^2) \xrightarrow{\partial_0 = 0} 0.$$

$\Delta_3(S^2) = 0$, $\Delta_2(S^2) \cong \text{span}\{U, L\}$, $\Delta_1(S^2) \cong \text{span}\{a, b, c\}$, $\Delta_0(S^2) \cong \text{span}\{u, v, w\}$.

$$\partial_1(a) = v - u$$
$$\partial_1(b) = w - v \qquad \text{Im}\,\partial_1 = \text{span}\{v-u, w-v, w-u\}$$
$$\partial_1(c) = w - u. \qquad = \text{span}\{w-v, w-u\}.$$

$$H_0^\Delta(S^2) = \text{span}\{u, v, w\} \Big/ \text{span}\{w-v, w-u\}$$
$$\simeq \mathbb{Z}.$$

$$\partial_1(pa + qb + rc) = (-p-r)u + (p-q)v + (q+r)w = 0.$$
only if $p = q = -r$.

$\therefore \ker \partial_1 = \text{span}\{a + b - c\}$.

$\partial_2(U) = b - c + a = \partial_2(L)$.

$\therefore \text{Im}\,\partial_2 = \ker \partial_1$

$$H_1^\Delta(S^2) = 0.$$

$\partial_2(pU + qL) = (p+q)(b - c + a) = 0$ only if $q = -p$.

$\therefore \ker \partial_2 = \text{span}\{U - L\}$.

$$H_2^\Delta(S^2) = \ker \partial_2 \simeq \mathbb{Z}$$

$$H_n^\Delta(S^2) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$