Singular Homology

- A singular n-simplex in a space X is by def. just a map $\sigma : \Delta^n \to X$. (its image does not look at all like a simplex.)
- Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X.
- Elements of $C_n(X)$, called (singular) n-chains, are finite formal sums $\sum_i n_i \sigma_i$, $n_i \in \mathbb{Z}$, and $\sigma_i : \Delta^n \to X$.
- A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by $\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ where $\sigma|[v_0, \cdots \hat{v}_i \cdots v_n]$ is regarded as a map $\Delta^{n-1} \to X$, i.e. a singular $(n-1)$-simplex.

- As seen in Lemma 2.1, one can show that $\partial_n \partial_{n+1} = 0$ so we can define the singular homology group

$$H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}.$$

Prop 2.6. Let $X = \coprod_\alpha X_\alpha$ be the decomposition of X into its path-components. Then

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Pf: $\because$ A singular simplex always has path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$.

The boundary maps ∂_n preserve this direct sum decomposition, taking $C_n(X_\alpha)$ into $C_{n-1}(X_\alpha)$,

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so $\ker \partial_n$ and $\operatorname{Im} \partial_{n+1}$ split similarly as direct sums, hence $H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha)$. #.

<u>prop 2.7</u> If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. Hence, for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.

<u>Pf</u>: $\cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0. \Rightarrow H_0(X) = C_0(X) / \operatorname{Im}\partial_1.$

Define $\varepsilon : C_0(X) \to \mathbb{Z}$ by $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$

This is obviously surjective. if X is nonempty.

<u>Claim</u>: $\ker \varepsilon = \operatorname{Im} \partial_1$. if X is path-connected.

By the claim: $H_0(X) = C_0(X)/\operatorname{Im}\partial_1 = C_0(X)/\ker\varepsilon$

$$\cong \operatorname{Im}\varepsilon = \mathbb{Z}. \quad \#.$$

<u>Pf of Claim</u>:

For any singular 1-simplex $\sigma : \Delta^1 \to X$, we have $\varepsilon \partial_1 \sigma = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0.$

Therefore, $\operatorname{Im}\partial_1 \subseteq \ker\varepsilon$.

Suppose $\sum_i n_i \sigma_i \in \ker\varepsilon$, then $\sum_i n_i = 0$.

The σ_i's are singular 0-simplices, which are simply points of X.

Choose a path $\tau_i : I \to X$ from a basepoint x_0 to $\sigma_i(v_0)$

$$x_0 \longrightarrow \sigma_i(v_0)$$

Let σ_0 be the singular 0-simplex with image x_0.
We can view τ_i as a singular 1-simplex,
a map $\tau_i : [v_0, v_1] \to X$,

$$\Rightarrow \partial \tau_i = \sigma_i - \sigma_0 \qquad \tau_i(v_0)=x_0,\ \tau_i(v_1) = \sigma_i(v_0).$$

Hence $\partial\left(\sum_i n_i \tau_i\right) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i$

$(\because \sum n_i = 0)$

Thus $\sum_i n_i \sigma_i$ is a boundary.

$\Rightarrow \operatorname{Ker}\varepsilon \subset \operatorname{Im}\partial_1.$ #.

<u>Prop 2.8</u> If X is a point, then $H_n(X) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n>0. \end{cases}$

<u>Pf:</u>

$\forall n \geq 0$, $\exists !$ singular n-simplex $\sigma_n : \Delta^n \to X$.

Therefore $C_n(X) \approx \mathbb{Z}$.

Notice that

$$\partial(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n \Big|_{[v_0, \cdots, \hat{v}_i \cdots v_n]}$$
$$= \sum_{i=0}^{n} (-1)^i \sigma_{n-1}$$
$$= \begin{cases} \sigma_{n-1} & n \text{ is even.} \\ 0 & n \text{ is odd.} \end{cases}$$

Therefore $\partial_n : C_n(X) \to C_{n-1}(X)$ is an isomorphism when n is even.
and the zero homomorphism when n is odd.
Therefore the singular chain complex of X is isomorphic to

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow[0]{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

$n>0$, $H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1} = \begin{cases} 0/0, & n \text{ even} \\ C_n(X)/C_n(X), & n \text{ odd} \end{cases} = 0.$

$n=0$. $H_0(X) = \mathbb{Z}$. #

Reduced Homology groups

Define the reduced homology groups $\tilde{H}_n(X)$ to be the homology groups of the augmented chain complex.

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$ surjective.

Since $\varepsilon \partial_1 = 0$, ε vanishes on $\operatorname{Im}\partial_1$.

$\rightsquigarrow$ a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$.

so $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.

Obviously $H_n(X) \simeq \tilde{H}_n(X)$ for $n > 0$.

$$C_0(X) \simeq \operatorname{Ker}\varepsilon \oplus \mathbb{Z}.$$

$$H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1} \simeq \underset{\displaystyle \frac{\operatorname{Ker}\varepsilon}{\operatorname{Im}\partial_1}}{\underset{\parallel}{\tilde{H}_0(X)}} \oplus \mathbb{Z}$$

$$\frac{\operatorname{Ker}\varepsilon}{\operatorname{Im}\partial_1} \oplus \mathbb{Z}.$$