# Review on point-set topology.

Point-set topology is concerned with "properties" that remain invariant under "homeomorphisms" (conti. maps having conti. inverses)

Defi: A <u>topological space</u> is a set $X$ together with a collection $\mathcal{T}$ of subsets of $X$, called "open sets", which satisfy.

(1) $X$ and $\emptyset$ are open.

(2) Any union of open sets is open

i.e. $U_\alpha \in \mathcal{T}$, for all $\alpha$ in an index set $A$

$\Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

(3). The intersection of any finite number of open sets is open.

i.e. $U_1, \cdots, U_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

- The collection $\mathcal{T}$ is called a <u>topology</u> on $X$
- The elements of $\mathcal{T}$ are called <u>open sets</u>.
- The pair $(X, \mathcal{T})$ is called a <u>topological space</u>
- The complement of an open set is called a closed set.

<u>Examples</u>

1. Let $X$ be a set.

Then $\mathcal{T} = \{\emptyset, X\}$ is a topology, called the <u>trivial</u> or <u>indiscrete</u> topology.

2. Let $X$ be a set.
Then $\mathcal{T} = \{$ all subsets of $X \}$ is a topology, called the <u>discrete</u> topology.

3. (<u>Metric topology</u>)
$\mathbb{R}^n$ with the standard topology (Advanced Calculus)
$U \subset \mathbb{R}^n$ open. $\Longleftrightarrow$ $\forall p \in U$, $\exists$ open ball $B(p, \varepsilon)$ with center $p$, radius $\varepsilon$ contained in $U$.

<u>Exercise</u>: $X$: a topological space.
$U$ open in $X$. $\Longleftrightarrow$ $\forall p \in U$, $\exists$ an open set $V$ s.t. $p \in V \subset U$.

$$B(p, \varepsilon) = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid \underbrace{\|x - p\|}_{= d(x,p)} = \sqrt{(x_1-p_1)^2 + \cdots + (x_n-p_n)^2} < \varepsilon \}$$

metric : $\underline{d(x, p)}$ Euclidean distance.

Let $(X, d)$ be a metric space.
$\mathcal{T} = \{$ subsets of $X$ that are unions of open balls $B(p, \varepsilon)$ in $X \}$

$$B(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

$\mathcal{T}$ is a topology, called the <u>metric topology</u>.

<u>Def</u>: A map $f : X \to Y$ between topological spaces $X, Y$ is <u>continuous</u> if $V \subset Y$ is open, then $f^{-1}(V) \subset X$ is open.

The idea behind the definition of a topological space $X$ is to discard all these properties of $X$ that have nothing to do with continuous maps.

Prop: If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

It's generally difficult to describe directly all the open sets in a topology $\mathcal{T}$. What one can usually do is to describe a subcollection $\mathcal{B}$ of $\mathcal{T}$ such that any open set is expressible as a union of open sets in $\mathcal{B}$.

Defi: A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ on a topological space $X$ is a basis for $\mathcal{T}$ if given an open $U$ & $p \in U$, there is an open set $B \in \mathcal{B}$ s.t. $p \in B \subset U$.

We also say that $\mathcal{B}$ generates the topology $\mathcal{T}$ ($\mathcal{B}$ is a basis for the topo. space $X$)

subspace topology

Defi: $X$: a topological space. $A \subset X$ a subset. Then $\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}$ is a topology of $A$ called the subspace topology (relative topology of $A$ in $X$)

$(A, \mathcal{T}_A)$ is a subspace of $X$.

Ex: Consider $A = [0,1] \subset \mathbb{R}^1$

$[0, \frac{1}{2})$ is open relative to $A$.

$\because [0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap A$

Prop: $A \subset X$ a subspace, $f: X \to Y$ continuous the restriction of $f$ to $A$ is defined by $(f|_A)(a) = f(a)$. $a \in A$.

If $i: A \to X$ inclusion map then $f|_A = f \circ i$ is continuous.

product topology:

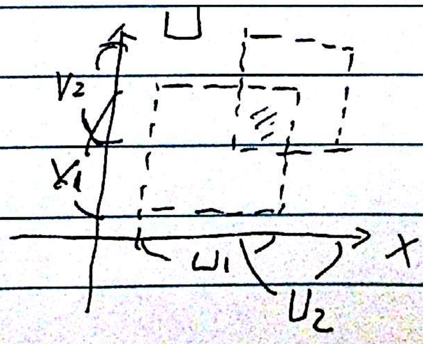
Def: The product topology on the Cartesian product $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ of topological spaces $X, Y$. is the topology with basis

$$\mathcal{B} = \{ U \times V \mid U \underset{\text{open}}{\subset} X,\ V \underset{\text{open}}{\subset} Y \}$$

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

Note: The collection $\mathcal{B}$ is not a topology.

$\because$ $(U_1 \times V_1) \cup (U_2 \times V_2)$ is typically not a Cartesian product.



## Hausdorff space

**Def:** A topological space $X$ is Hausdorff if $\forall\, x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

**Ex:** If $X$ is a metric space, then the metric topology on $X$ is Hausdorff.
( $\because x \neq y$, choose $\varepsilon = \frac{d(x,y)}{2}$.
$B(x, \varepsilon)$ & $B(y, \varepsilon)$ are disjoint open subsets.

**Ex:** The line with two origins:
$(\mathbb{R} \times \{0,1\})/\sim$ , $(x,0) \sim (x,1)$ for $x \neq 0$.
Not Hausdorff. $\because$ we cannot separate $(0,0)$ and $(0,1)$.

**Def:** $x_i \in X$, $i = 1, 2, \cdots$ a sequence in $X$ and $x \in X$. $\Rightarrow \lim_{i \to \infty} x_i = x$ if for any open subset $U \subset X$, $x \in U$.
$\exists N \in \mathbb{N}$ s.t. $x_i \in U$, $\forall i \geq N$.

The limit is unique if $X$ is Hausdorff.

**Def:** An open cover $\{U_\alpha\}$ of a topo. space $X$ is a collection of open subsets of $X$ whose union is $X$. i.e. $U_\alpha \in X$, $\bigcup_\alpha U_\alpha \supset X$.

If for every open cover of $X$, there is a finite subcover which also covers $X$, then $X$ is called <u>compact</u>.

<u>Prop</u> (1) If $f: X \to Y$ is conti. & $X$ is compact, then $f(X)$ is compact.

(2). If $K$ is a closed subset of a cpct top. space $X$, then $K$ is compact.

(3) If $K$ is a cpct subset of a Hausdorff space $X$, then $K$ is closed.

(4). If $f: X \to Y$ is a continuous bijection with $X$ cpct & $Y$ Hausdorff, then $f$ is a homeomorphism.

<u>Theorem</u> (<u>The Tychonoff theorem</u>).
The product of any collection of compact spaces is compact in the product topo.

<u>Boundedness in $\mathbb{R}^n$</u>

<u>Def</u>: A subset $A$ of $\mathbb{R}^n$ is called <u>bounded</u> if $A \subset B(p, r)$ some open ball.

<u>Theorem</u> (<u>Heine-Borel theorem</u>)
A subset of $\mathbb{R}^n$ is cpct. $\Longleftrightarrow$ It is closed and bounded.

## Connectedness

Def: A topo. space $X$ is disconnected if $X = U \cup V$ of two disjoint nonempty open subsets $U$ and $V$.

It's connected if it is not disconnected.

Prop: The image of a connected space $X$ under a continuous map $f: X \to Y$ is connected.

Def: A topo. space $X$ is path-connected if for any points $x, y \in X$, there is a path connecting them.
i.e. There is a conti. map $\gamma : [a,b] \to X$ with $\gamma(a) = x$, $\gamma(b) = y$.

Prop: Any path-connected topo. space is connected.

Example: Topologist's sine curve which is connected, but not path-connected

$$X = \{(x, \sin\tfrac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1\} \cup \{(0,y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}$$

Quotient topology

Defi $X$: a topo. space.

$\sim$ : an equivalence relation on $X$.

Denote by $X/\sim$ the set of equivalence classes.

and by $\pi : X \to X/\sim$

$x \longrightarrow [x]$.

The quotient topology on $X/\sim$ is given by the collection of subsets

$$\mathcal{T} = \{ U \subset X/\sim \mid \pi^{-1}(U) \overset{\text{open}}{\subset} X \}$$

The set $X/\sim$ equipped with the quotient topology is called the quotient space.

suppose $f : X \to Y$ is constant on each equivalence class.

$\Rightarrow$ It induces a map $\bar{f} : X/\sim \to Y$ by

$$\bar{f}([p]) = f(p) \qquad p \in X.$$

i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & Y \\ X/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Prop: The induced map $\bar{f} : X/\sim \to Y$ is conti.

$\Longleftrightarrow$ $f : X \to Y$ is conti.

Pf: ($\Rightarrow$) $\bar{f}$ conti. $\Rightarrow$ $\bar{f} \circ \pi$ conti. $\Rightarrow$ $f$ conti.

($\Leftarrow$). Suppose $f$ conti.

Let $V$ open in $Y$. $\Rightarrow$ $f^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ open in $X$.

By def. of quotient topo. $\bar{f}^{-1}(V)$ open in $X/\sim$

$\because$ $V$ is arbitrary.

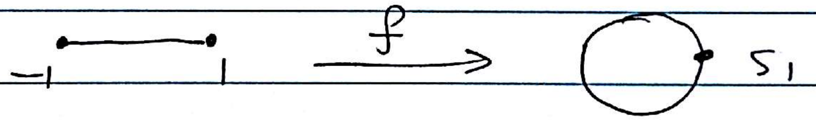
$\Rightarrow \bar{f} : X/\sim \to Y$ conti. #.

<u>Example</u> (1). Consider $[-1,1]/\sim$, $-1 \sim +1$.

$f : [-1,1] \to S^1$

$x \to \exp(\pi i x)$.

$\rightsquigarrow \bar{f} : [-1,1]/\sim \to S^1$.




<u>Prop:</u> $\bar{f} : [-1,1]/\sim \to S^1$ a homeomorphism.

<u>Pf:</u> $f$ conti. $\xrightarrow{\text{Prop}}$ $\bar{f}$ conti.

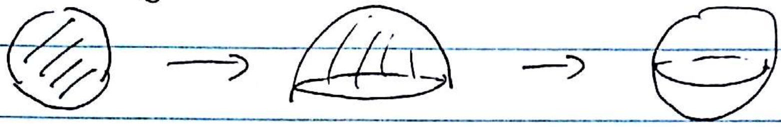
Clearly $\bar{f}$ is a bijection.

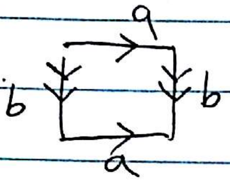
$[-1,1]$ cpct. $\Rightarrow$ $\underbrace{[-1,1]/\sim}_{\pi([-1,1])}$ cpct.

$\bar{f}$ : a conti. bijection from cpct space $[-1,1]/\sim$ to the Hausdorff space $S^1$

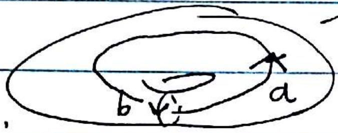
$\Rightarrow \bar{f}$ is a homeomorphism. #.

(2). More generally. $D^n/S^{n-1}$ is homeomorphic to $S^1$.

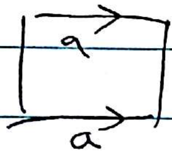


(3). 

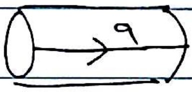

$[-1,1]\times[-1,1]/\sim$. $(s,-1)\sim(s,1)$, $s,t\in[-1,1]$.
$(-1,t)\sim(1,t)$

$\underset{\text{homeom.}}{\simeq}$ 

torus.

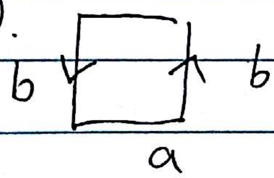
(4) 

$= [-1,1]\times[-1,1]/(s,-1)\sim(s,+1)$.

$\simeq$ 

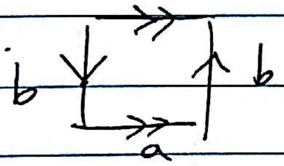
cylinder $C=\{(x,y,z)\in\mathbb{R}^3 \mid x\in[-1,1],\ y^2+z^2=1\}$.

$f([s,t])=(s,\ \sin\pi t,\ \cos\pi t)$

(5). 

$[-1,1]\times[-1,1]/\sim$ $(-1,t)\sim(1,-t)$.

Möbius band.

(6). 

Klein bottle

(7). Real projective space. $\mathbb{R}P^n$

$\mathbb{R}P^n := \mathbb{R}^{n+1}-\{0\}/\sim$. $x\sim y \Leftrightarrow y=tx$, $t\in\mathbb{R}\setminus\{0\}$.
$x,y\in\mathbb{R}^{n+1}-\{0\}$.

$\simeq S^n/\sim$. $x\sim y \Leftrightarrow x=\pm y$. $x,y\in S^n$.

$\simeq D^n/\sim$ $v\sim -v$, $v\in S^{n-1}\subset D^n$.

In particular, $\mathbb{R}P^1 = S^1/\sim \simeq D^1/\sim \simeq S^1$

$\mathbb{R}P^2$.