Reidemeister torsion

References:
1. Notes on Reidemeister torsion, Andrew Ranicki
2. Introduction to combinatorial torsions, V. Turaev.

Some traditional algebraic topology invariants of a finite simplicial complex X:
- the Betti numbers $b_*(X)$ (1871)
- the fundamental group $\pi_1(X)$ (1895)
- the homology groups $H_*(X)$, $b_i(X) = \dim H_i(X)$ (1925).

1935. Reidemeister introduced Reidemeister torsion. (R-torsion) on the combinatorial classification of the 3-dim$\hat{l}$ len spaces by means of the based simplicial chain complex of the universal cover.

1935. Franz (a student of Reidemeister) generalized to higher dim$\hat{l}$ case.

R-torsion:
- a combinatorial invariant: finite simplicial complexes with isomorphic subdivisions have same R-torsion.
- not a homotopy invariant: $\exists$ homotopy equivalent spaces with different R-torsion.
- a topological invariant: finite simplicial complexes with homeomorphic polyhedra have the same R-torsion.

Torsion of chain complexes

- Let D be a finite-dim'l vector space over a field $\mathbb{F}$. $\dim D = k$.

Pick two (ordered) bases $b = (b_1, \cdots, b_k)$ and $c = (c_1, \cdots, c_k)$ of D.

Then $b_i = \sum_{j=1}^{k} a_{ij} c_j \quad i = 1, \cdots, k.$

the matrix matrix $(a_{ij})_{i,j=1,\cdots,k}$ is a non-degenerate $(k \times k)$-matrix over F.

Write $[b/c] = \det(a_{ij}) \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$.

Clearly, (1) $[b/b] = 1$.

(2). If d is a third basis of D, then $[b/d] = [b/c] \cdot [c/d]$.

We call two bases b and c <u>equivalent</u> $(b \sim c)$ if $[b/c] = 1$.

$\sim$ is an equivalence relation.

- Let $0 \to C \hookrightarrow D \xrightarrow{\beta} E \to 0$ be a short exact sequence of vector spaces.

Then $\dim D = \dim C + \dim E$.

Let $c = (c_1, \cdots, c_k)$ be a basis of C.

$e = (e_1, \cdots, e_\ell)$ be a basis of E.

Since β is surjective, we may lift each e_i to some $\tilde{e}_i \in D$.

Set $ce = (c_1, \cdots, c_k, \tilde{e}_1, \tilde{e}_2 \cdots \tilde{e}_\ell)$.

Then ce is a basis of D.
Its equivalence class does not depend on the choice of $\tilde{e}_i$. It depends only on the equivalence classes of c and e.

Now we consider the chain complex

$$C = (0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0)$$

where $C_0, C_1, \ldots, C_m$ are finite-dim'l v.s. / $\mathbb{F}$.

Def: The chain complex C is ① acyclic if $H_i(C) = 0$, $\forall i$.
(i.e. $\operatorname{Ker} \partial_{i-1} = \operatorname{Im} \partial_i$, $\forall i$.)
② based if each C_i has a distinguished basis c_i.

Let $C = (0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0)$ be an acyclic based chain complex over $\mathbb{F}$.
Set $B_i = \operatorname{Im}(\partial_i : C_{i+1} \to C_i) \subset C_i$.
Since C is acyclic.

$$C_i / B_i = C_i / \operatorname{Ker}(\partial_{i-1} : C_i \to C_{i-1}) \cong \operatorname{Im} \partial_{i-1} = B_{i-1}.$$

In other words,

$$0 \to B_i \hookrightarrow C_i \xrightarrow{\partial_{i-1}} B_{i-1} \to 0$$

is exact.
Choose a basis b_i of B_i for $i = -1, \ldots, m$. ($B_{-1} = 0$, $B_m = 0$)
$b_i b_{i-1}$ is a basis of C_i.

Def: The torsion of C is $\tau(C) = \prod_{i=0}^{m} \left[\frac{b_i b_{i-1}}{c_i} \right]^{(-1)^{i+1}} \in \mathbb{F}^*$.

- $\tau(C)$ does not depend on the choice of b_i but depends on the distinguished basis c_i of C_i.

Computation of the torsion.

Chain contraction

Fix an acyclic based finite dim'l chain complex
$$C = (0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_0} C_0 \to 0)$$
over a field $\mathbb{F}$.

Since C is acyclic, $\exists$ a <u>chain contraction</u> $\delta: C \to C$

i.e. a sequence of homomorphisms
$$\delta_i : C_i \to C_{i+1},\ i = 0, 1, \ldots, m \text{ s.t. for all } i,$$
$$\delta_{i-1}\partial_{i-1} + \partial_i \delta_i = Id : C_i \to C_i.$$
(a standard result of homological algebra.)

Set $B_i = \mathrm{Im}(\partial_i : C_{i+1} \to C_i)$.

Then the short exact sequence
$$0 \to B_i \hookrightarrow C_i \xrightarrow{\partial_{i-1}} B_{i-1} \to 0$$
splits as $C_i = B_i \oplus \sigma_i(B_{i-1})$, where $\partial_{i-1} \circ \sigma_i = Id$.

(Since we work over a field, the splitting lemma, p147, Hatcher implies the result.)

Define $\delta_i : C_i = B_i \oplus \sigma_i(B_{i-1}) \to C_{i+1}$

by $\delta_i(a+b) = \sigma_{i+1}(a)$ where $a \in B_i$, $b \in \sigma_i(B_{i-1})$.

For $b = \sigma_i(b')$, $b' \in B_{i-1}$, we have
$$(\delta_{i-1}\partial_{i-1} + \partial_i\delta_i)(a+b) = \delta_{i-1}(\partial_{i-1}b) + \partial_i(\sigma_{i+1}(a))$$
$$= \delta_{i-1}(b') + a = \sigma_i(b') + a = b + a = \underline{a+b}.$$

Thus $\delta_{i-1}\partial_{i-1} + \partial_i\delta_i = Id$.

Set $C_{even} = \bigoplus_{i \text{ even}} C_i$.

$C_{odd} = \bigoplus_{i \text{ odd}} C_i$.

Clearly $\partial + \delta$ maps C_{even} to C_{odd}
and C_{odd} to C_{even}.

<u>Theorem</u>. For any chain contraction δ,

$$\tau(C) = \det(\partial + \delta : C_{even} \to C_{odd}) = \left(\det(\partial + \delta : C_{odd} \to C_{even})\right)^{-1}$$

<u>Remark</u>: Since C_{even} and C_{odd} are based
and since $\dim C_{even} - \dim C_{odd} = \chi(C) = 0$.
It makes sense to consider determinants
of $\partial + \delta$.

<u>Formulation of Ray-Singer.</u>

suppose $\mathbb{F} = \mathbb{R}$,
Then there is a unique Euclidean metric $\langle \cdot, \cdot \rangle_i$
on C_i s.t. the distinguished basis c_i is an
orthonormal basis.

Define the adjoint $\partial_i^* : C_i \to C_{i+1}$ by

$$\langle \partial_i^* a, b \rangle_{i+1} = \langle a, \partial_i b \rangle_i . \quad a \in C_i,\ b \in C_{i+1}.$$

and form the Laplacian

$$\Delta_i = \partial_{i-1}^* \partial_i + \partial_i \partial_i^* : C_i \to C_i .$$

$|\tau(C)| \in \mathbb{R}_+$ can be recovered from the spectra
of Δ_i, $i = 0, \dots, m$, via an explicit formula.

<u>Remark</u>. $\mathbb{F}$ can be replaced by a <u>commutative ring</u> A.
C_i " " " " a <u>A-module</u>. finitely generated free.
$\tau(C) \in A^*$.

How does one associate an acyclic chain complex to a space X?

Consider the CW decomposition $X = \bigcup_{r=0}^{\infty} \cup e^r$.

Lift each cell $e^r \subset X$ to a cell $\tilde{e}^r \subset \tilde{X}$ in the universal cover of X.

$$\tilde{X} = \bigcup_{g \in \pi_1(X)} \bigcup_{r=0}^{\infty} \cup g\tilde{e}^r$$

with $\pi_1(X)$ acting on $\tilde{X}$ as the group of covering translations $\pi_1(X) \times \tilde{X} \to \tilde{X}$

$(g, x) \mapsto gx$

The group ring $\mathbb{Z}[\pi_1(X)]$ consists of the finite linear combinations $\sum_{g \in \pi_1(X)} n_g g \ (n_g \in \mathbb{Z})$

The cellular chain complex of $\tilde{X}$

$$C(\tilde{X}) : \ \dashrightarrow C(\tilde{X})_{r+1} \xrightarrow{\partial} C(\tilde{X})_r \xrightarrow{\partial} C(\tilde{X})_{r-1} \to \cdots \to C(\tilde{X})_0$$

is a chain complex of based free $\mathbb{Z}[\pi_1(X)]$-modules, with $C(\tilde{X})_r = H_r(\tilde{X}^{(r)}, \tilde{X}^{(r-1)})$

$=$ based f.g. free $\mathbb{Z}[\pi_1(X)]$-module generated by the r-cells $e^r \subset X$

with $\tilde{X}^{(r)}$ the induced cover of the r-skeleton of X. $X^{(r)} = \bigcup_{j \leq r} \cup e^j \subseteq X$.

The basis elements are only determined by the cell structure of X up to multiplication by $\pm g$ $(g \in \pi_1(X))$.

The geometric action of $\pi_1(X)$ on $\tilde{X}$ induces the algebraic action of $\mathbb{Z}[\pi_1(X)]$ on $C(\tilde{X})$.

Now $H_0(C(\tilde{X})) = H_0(\tilde{X}) = \mathbb{Z}$.

so $C(\tilde{X})$ is <u>not</u> acyclic & does <u>not</u> have a Reidemeister torsion.

<u>Trick:</u> Find a commutative ring A with a ring morphism $f: \mathbb{Z}[\pi_1(X)] \to A$ s.t. the induced A-module chain complex

$$C(X;A) = A \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X})$$

is acyclic.

<u>Def:</u> The Reidemeister torsion of a finite CW complex X w.r.t. a ring morphism $f: \mathbb{Z}[\pi_1(X)] \to A$ s.t. $H_*(X;A) = 0$ is

$$\tau(X) = \tau(C(X;A)) \in A^\bullet / \{f(\pm \pi_1(X))\}$$

<u>Def:</u> (Tietze, 1908) The <u>lens spaces</u> are the closed oriented 3-dim'l manifolds

$$L(m,n) = S^3/\mathbb{Z}_m = \{(a,b) \in \mathbb{C}\times\mathbb{C} \mid |a|^2+|b|^2=1\} \Big/ (a,b) \sim (\xi a, \xi^n b)$$

with $\xi = e^{\frac{2\pi i}{m}}$ a primitive m-th root of unity, and m, n coprime.

Standard homotopy invariants of $L = L(m,n)$ are $\pi_1(L) = H_1(L) = \mathbb{Z}_m$, $\pi_i(L) = \pi_i(S^3)$, $i \geq 2$.

$H_0(L) = H_3(L) = \mathbb{Z}$, $H_i(L) = 0$, $i \neq 0,1,3$.

$b_0(L) = b_3(L) = 1$, $b_i(L) = 0$, $i \neq 0,3$.

Example: $L(1,1) = S^3$, $L(2,1) = \mathbb{RP}^3$.

Let $X = L(m,n)$. and choose a generator $t \in \pi_1(X) = \mathbb{Z}_m$

$$\mathbb{Z}[\mathbb{Z}_m] = \mathbb{Z}[t, t^{-1}]/(1-t^m).$$

$\because$ m, n coprime, $\exists\, a, b \in \mathbb{Z}$ s.t. $an + bm = 1$.

$L(m,n)$ has a CW decomposition.

$$L(m,n) = e^0 \cup e^1 \cup e^2 \cup e^3.$$

which lifts to a $\mathbb{Z}_m$-equivariant CW structure on the universal cover

$$\widetilde{L(m,n)} = S^3 = \bigcup_{k=0}^{m-1} \left(t^k \tilde{e}^0 \cup t^k \tilde{e}^1 \cup t^k \tilde{e}^2 \cup t^k \tilde{e}^3 \right).$$

The cellular chain complex of based f.g. free $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$C = C(\widetilde{L(m,n)}) : \cdots \to 0 \to \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{1-t^a} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_m]$$

with $N = 1 + t + \cdots + t^{m-1} \in \mathbb{Z}[\mathbb{Z}_m]$.

For each primitive mth root ζ of 1 in $\mathbb{C}$

$\exists$ defined ring morphism $f_\zeta : \mathbb{Z}[\mathbb{Z}_m] \to \mathbb{C}$

$$t \longmapsto \zeta.$$

s.t. $H_*(L(m,n);\mathbb{C})=0$ $(\because 1+\zeta+\zeta^2+\cdots+\zeta^{m-1}=0)$.

with Reidemeister torsion given by

$$\tau(C(L(m,n);\mathbb{C}))=(1-\zeta)(1-\zeta^q)\in \mathbb{C}^\bullet/\{f(\pm\mathbb{Z}_m)\}.$$

- The Reidemeister torsion of $L(m,n)$ fishes out the topologically relevant part of n from $L(m,n)$

Theorem (Franz, Rueff, Whitehead (1940))

(i) The following conditions are equivalent:
- $L(m,n)$ is homotopy equivalent to $L(m,n')$
- $n\equiv \pm n' r^2 \pmod m$ for some $r\in\mathbb{Z}_m^\bullet$.

(ii). The following conditions are equivalent:
- $L(m,n)$ is homeomorphic to $L(m,n')$.
- $n\equiv \pm n' r^2 \pmod m$ with $r\equiv 1$ or $n \pmod m$,
- ....

Example: $L(5,1)$ is not homotopy equivalent to $L(5,2)$.

Example: $L(7,1)$ is homotopy equivalent but not homeomorphic to $L(7,2)$.