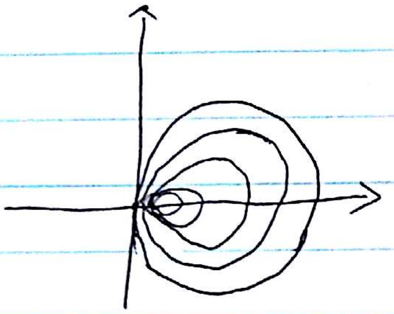
The classification of covering spaces

A space X is said to be <u>semilocally simply-connected</u> if for each point $x \in X$, there is an open neighborhood U such that the inclusion-induced map $\pi(U, x) \to \pi_1(X, x)$ is trivial.

- <u>The shrinking wedge of circles (Example 1.25)</u>

$$X = \bigcup_{i=1}^{\infty} \left\{ (x, y) \,\middle|\, \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}$$



is <u>not</u> semilocally simply-connected.

. But the cone $CX = \dfrac{X \times I}{X \times \{0\}}$ is semilocally simply-connected since $\pi_1(CX) = 0$. i.e. CX is contractible.

Note that CX is <u>not</u> locally simply-connected.

<u>Recall:</u> A space X is said to be <u>locally simply-connected</u> if for every $x \in X$, and every open subset V of X containing x, there is an open subset U of X contained in V, and which is simply connected in the subspace topology from X.

- A locally semi-connected space is certainly semilocally simply-connected.

- Let X be a space which is connected and locally path-connected. Then X is path-connected.

Pf:

Fix $a \in X$. and let $C = \{x \in X : \exists$ a path in X from a to $x.\}$.

We want to show that C is open and closed.

The assumption that X is locally path-connected implies that C is open.

suppose $x \notin C$ and choose a path-connected open set U containing x.

if $y \in U$, then there is a path h in U from y to x.

Therefore there cannot exist a path in X from a to y. (why?)

$\Rightarrow y \notin C. \Rightarrow x \in U \subseteq X - C \Rightarrow C$ is closed. ※.

- suppose that $p : \tilde{X} \to X$ is a <u>universal</u> covering space. Then X is semilocally simply-connected.

$\pi_1(\tilde{X}) = 0$ + path connected

Pf:

Every point $x \in X$ has a neighborhood U having a lift $\tilde{U} \subset \tilde{X}$ projecting <u>homeomorphically</u> to U by p.

Each loop in U lifts to a loop in $\tilde{U}$ & the lifted loop is nullhomotopic in $\tilde{X}$ since $\pi_1(\tilde{X}) = 0$

So composing ~~this~~ nullhomotopy with p, the original loop in U is nullhomotopic in X. ※.

• If X is path-connected, locally path-connected and semilocally simply-connected, then there is a universal covering space $p: \tilde{X} \to X$.

pf:

We construct $\tilde{X}$ as follows:

Let $x_0 \in X$ and define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0\}.$$

where $[\gamma]$ denotes the homotopy class of γ w.r.t. homotopies that fix $\gamma(0)$ and $\gamma(1)$.

$p: \tilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$ is well-defined since homotopic paths have the same endpoints.

p is surjective since X is path-connected.

To define a topology on $\tilde{X}$, we choose a basis for the topology on X first.

Let $\mathcal{U} = \{ U \subseteq X \mid U \text{ is open and path-connected}, \ \pi_1(U) \to \pi_1(X) \text{ is trivial} \}$.

Suppose $x \in U_1 \cap U_2$ for some $U_1, U_2 \in \mathcal{U}$.

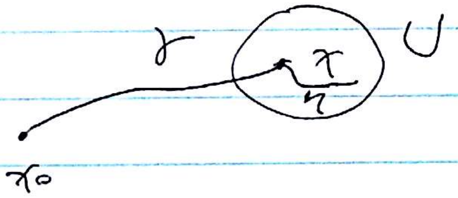
For any path-connected open neighborhood U of x satisfying $U \subseteq U_1 \cap U_2$.

$\pi_1(U) \to \pi_1(X)$ is trivial since it is the composite $\pi_1(U) \to \pi_1(U_i) \to \pi_1(X)$.

There $U \in \mathcal{U}$ and hence $\mathcal{U}$ is a basis. on X if X is locally path-connected and semilocally path-connected.

Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point $x \in U$

let $U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta$ is a path in U with $\eta(0) = \underbrace{\gamma(1)}_{= x} \}$



Notice that $U_{[\gamma]}$ depends only on the homotopy class of γ.

Since U is path-connected, $p : U_{[\gamma]} \to U$ is surjective

$[\gamma \cdot \eta] \longmapsto (\gamma \cdot \eta)(1)$

If $p([\gamma \cdot \eta_1]) = p([\gamma \cdot \eta_2])$ in U,
then $\eta_1 \cdot \overline{\eta_2}$ is a loop in U based at $\gamma(1)$

Since $\pi_1(U) \to \pi_1(X)$ is trivial, $\eta_1 \simeq \eta_2$ in X.

hence $\gamma \cdot \eta_1 \simeq \gamma \cdot \eta_2$.

This shows that $p : U_{[\gamma]} \to U$ is injective.

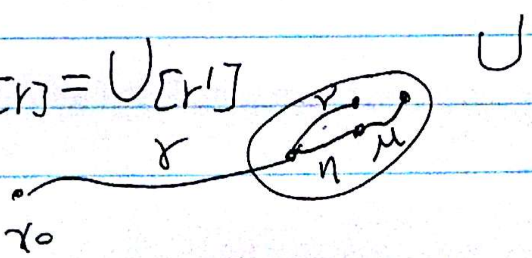
We show that the collection

$$\tilde{\mathcal{U}} = \{ U_{[\gamma]} \mid U \in \mathcal{U},\ \gamma(0) = x_0,\ \gamma(1) \in U \}$$

is a basis for a topology on $\tilde{X}$.

We first show that

(*) $[\gamma'] \in U_{[\gamma]} \Rightarrow U_{[\gamma]} = U_{[\gamma']}$



$[\gamma'] \in U_{[\gamma]}$

<u>Pf</u>: If $\gamma' \simeq \gamma \cdot \eta$, then $\gamma' \cdot \mu \simeq \gamma \cdot \underline{\eta \cdot \mu}$ for any path μ in U with $\mu(0) = \gamma'(1)$.

This shows that $U_{[\gamma']} \subseteq U_{[\gamma]}$.

On the other hand, for any path ν in U with $\gamma(1) = \nu(0)$, we have

$$[\gamma \cdot \nu] = [\gamma \cdot \eta \cdot \bar{\eta} \cdot \nu] = [\gamma' \cdot \underline{\bar{\eta} \cdot \nu}]$$

$$\Rightarrow \quad U_{[\gamma]} \subseteq U_{[\gamma']}. \qquad \divideontimes$$

If $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ for some $U, V \in \mathcal{U}$, then $(*)$ implies that $U_{[\gamma'']} = U_{[\gamma]}$

and $V_{[\gamma'']} = V_{[\gamma']}$.

So if $W \in \mathcal{U}$ such that $\gamma''(1) \in W$ and $W \subseteq U \cap V$. Then $W_{[\gamma'']} \subseteq U_{[\gamma]} \cap V_{[\gamma']}$ and $[\gamma''] \in W_{[\gamma'']}$.

This shows that $\tilde{\mathcal{U}}$ is a basis for a topology on $\tilde{X}$.

The bijection $p : U_{[\gamma]} \to U$ is a homeomorphism, since for any $V \in \mathcal{U}$ with $V \subset U$ and $[\gamma'] \in U_{[\gamma]}$ with $\gamma'(1) \in V$, we have

$$p(V_{[\gamma']}) = V \subseteq U$$

and $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$.

$\left(V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}, \right.$ $V_{[\gamma']}$ maps onto V by p.

It follows that $p: \tilde{X} \to X$ is continuous and hence is a covering space.

Given any $[\gamma] \in \tilde{X}$, we define γ_t by

$$\gamma_t(s) = \begin{cases} \gamma(s), & 0 \le s \le t \\ \gamma(t), & t \le s \le 1. \end{cases}$$

Then $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ which lifts γ starting at $[x_0]$, where x_0 is the constant path in X at x_0.

~~This is a path from $[x_0]$ to $[\gamma]$ in $\tilde{X}$,~~ $\overset{[\gamma_1]}{=}$

Since $[\gamma]$ was an arbitrary point in $\tilde{X}$, ~~this~~ shows that $\tilde{X}$ is path-connected.

To show that $\pi_1(\tilde{X}, [x_0]) = 0$ it suffices to show that $p_*(\pi_1(\tilde{X}, [x_0])) = 0$ since $p_*: \pi_1(\tilde{X}, [x_0]) \to \pi_1(X, x_0)$ is injective.

If γ is a loop in X based at x_0, representing an element of $p_*(\pi_1(\tilde{X}, [x_0]))$, then it lifts to a loop in $\tilde{X}$ based at $[x_0]$.

~~Since $t \mapsto [\gamma_t]$ is a lift of γ based at $[x_0]$,~~

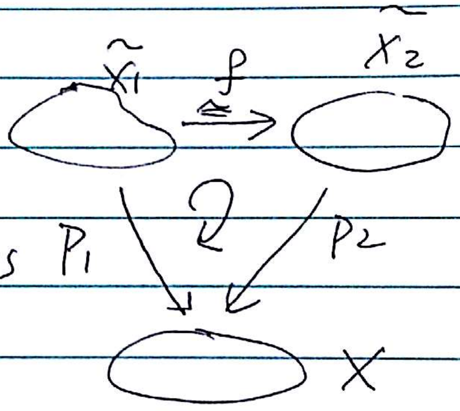
~~For this lifted~~ path to be a loop means that $[\gamma_1] = [x_0]$.

since $\gamma_1 = \gamma \Rightarrow [\gamma] = [x_0]$.

So γ is null homotopic and $P_*(\pi_1(\tilde{X}, [x_0])) = 0$.

This completes the construction of a simply-connected (universal) covering space $\tilde{X} \to X$. #

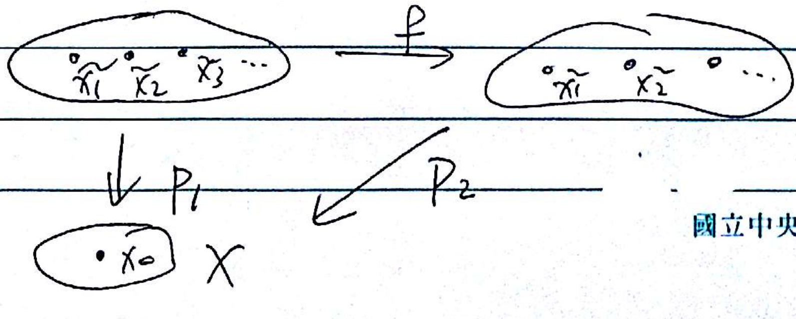
- An isomorphism between covering spaces $P_1 : \tilde{X}_1 \to X$ and $P_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $P_1 = P_2 f$



$\Rightarrow$ f^{-1} is also an isomorphism

The composition of two isomorphisms is an isomorphism, so we have an equivalence relation.

Prop 1.37: If X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ taking a basepoint $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p_2^{-1}(x_0)$ iff $P_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = P_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.



Recall: # of sheets of a covering space
$p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with X and $\tilde{X}$ path-connected
$=$ index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$
(prop 1.32).

Corollary: $\tilde{X}$ is simply-connected, $p: \tilde{X} \to X$ is a surjective covering space, then for any $x \in X$, there is a bijection
$$p^{-1}(x) \cong \pi_1(X, x).$$
"uniqueness up to isomorphism of covering spaces."

- For a covering space $p: \tilde{X} \to X$ the isomorphisms $\tilde{X} \to \tilde{X}$ are called <u>deck</u> <u>transformation</u>
(<u>covering</u>)

These form a group $G(\tilde{X})$ under composition.

<u>EX</u>: Consider the covering space $p: \mathbb{R} \to S^1$
$t \mapsto \exp(2\pi i t)$.
taking 1 as the base point.
$$p^{-1}(1) = \mathbb{Z}$$
the covering transformations $\mathbb{R} \xrightarrow{f} \mathbb{R}$ are maps $t \mapsto t+n$, $n \in \mathbb{Z}$. form a group isomorphic to $(\mathbb{Z}, +)$.
i.e. $G(\tilde{X}) \cong \mathbb{Z}$

EX: $p: S^1 \to S^1$, $\quad G(\tilde{X}) \cong \mathbb{Z}_n$
$z \mapsto z^n$

• Recall:
A covering space is a <u>universal</u> covering space if it is simply connected.

<u>Prop:</u> $p:(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is the universal covering space of the path-connected, locally path-connected space X, then

$$G(\tilde{X}) \simeq \pi_1(X).$$

<u>EX:</u> $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2$ for $n \geq 2$.

since $S^n \to \mathbb{R}P^n$ is the universal cover.

$(\pi_1(S^n) = 0)$ and $p^{-1}([x]) = \{x, -x\}$.