§1.2. Van Kampen's Theorem

Theorem 1.20

If X is the union of path-connected open sets A_α each containing the base point x_0 and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism

$$\Phi: *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective.

If in addition, each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$$

Notations:

- $*_\alpha \pi_1(A_\alpha)$ is the free product of the $\pi_1(A_\alpha)$'s.
- $i_{\alpha\beta}: \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.

Recall: the free product $*_\alpha G_\alpha$ of groups G_α consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length where $g_i \in G_{\alpha_i}$, $g_i \neq Id$, g_i, g_{i+1} belong to different groups G_α, i.e. $\alpha_i \neq \alpha_{i+1}$.

Example 1.21 Wedge sums

- Let X be the wedge sum $\bigvee_\alpha X_\alpha$ of path-connected spaces X_α with basepoint x_α, where all x_α's are identified to $x_0 \in X$.
- suppose x_α is a deformation retract of an open neighborhood U_α for all α. Then

$$A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$$

deformation retracts onto X_α and obviously $X = \bigcup_\alpha A_\alpha$

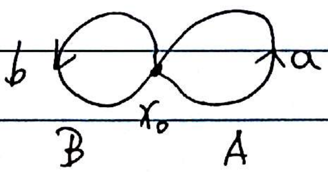
- Notice that the intersection of two or more of A_α's is the contractible space $\bigvee_\alpha U_\alpha$.
- The homomorphism

$$\Phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

is an isomorphism. $*_\alpha \pi_1(A_\alpha) \simeq \pi_1(X)$.

- ex: $\pi_1(\bigvee_\alpha S^1)$ is a free group, the free product of copies of $\mathbb{Z}$, one for each S^1_α.

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$$



$a^5 b^2 a^{-3} b a^2 \in \mathbb{Z} * \mathbb{Z}$

the loop that goes 5 times around A, then twice around B, then 3 times around A in opposite direction, then once around B, then twice around A.

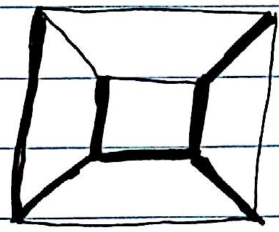
$(b^4 a^5 b^2 a^{-3})(a^4 b^{-1} a b^3)$
$= b^4 a^5 b^2 a b^{-1} a b^3$

The identity element is the empty word.

$(a b^2 a^{-3} b^{-4})^{-1} = b^4 a^3 b^{-2} a^{-1}$

Example 1.22. • Let X be the graph consisting of 12 edges of a cube.

• The 7 heavily shaded edges form a maximal tree $T \subset X$, (a contractible subgraph containing all the vertices of X.)



• Let A_α be an open neighborhood of $T \cup e_\alpha$ in X that deformation retracts onto $T \cup e_\alpha$.

• The intersection of 2 or more A_α's deformation retracts onto T, hence is contractible.

• By Van Kampen's theorem, we have

$$\begin{aligned}\pi_1(X) &\approx \pi_1(A_1) * \cdots * \pi_1(A_5)\\ &\simeq \mathbb{Z} * \cdots * \mathbb{Z}\\ &= \text{free group of rank } 5.\end{aligned}$$

$\because$ Each A_α deformation retracts onto a circle.

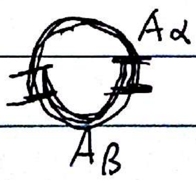
Remark: Van Kampen's theorem is often applied when there are just two sets A_α and A_β in the cover of X. Then one obtain

$$\pi_1(X) \approx (\pi(A_\alpha) * \pi_1(A_\beta))/N,$$

under the assumption that $A_\alpha \cap A_\beta$ is path-connected.

• $A_\alpha \cap A_\beta$ need to be path-connected.

Ex: $S^1 =$ union of two open arcs A_α, A_β



$A_\alpha \cap A_\beta = \text{I} \quad \text{I}$ not path-connected.

$\pi_1(A_\alpha) = \pi_1(A_\beta) = 0$. $\pi_1(S^1) = \mathbb{Z}$

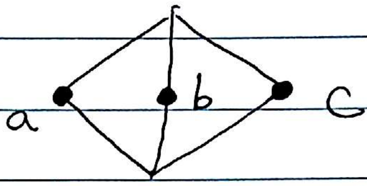
Φ is not surjective.

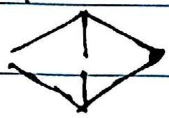
- $A_\alpha \cap A_\beta \cap A_\gamma$ need to be path-connected.

Ex: $A_\alpha = X - \{a\}$

$A_\beta = X - \{b\}$

$A_\gamma = X - \{c\}$

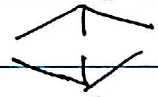
$X =$ 

$A_\alpha \cap A_\beta =$  path-connected. contractible.

$X = A_\alpha \cup A_\beta$.

By Van-Kampen's theorem, $\pi_1(X) \simeq \pi_1(A_\alpha) * \pi_1(A_\beta)$
$= \mathbb{Z} * \mathbb{Z}$

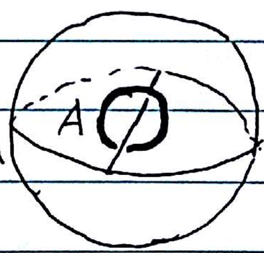
- If we try $X = A_\alpha \cup A_\beta \cup A_\gamma$, then $\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$
$\not\simeq \mathbb{Z} * \mathbb{Z}$

$\because$ $A_\alpha \cap A_\beta \cap A_\gamma = X - \{a, b, c\}$
$=$ 
not path-connected.

Example 1.23: Linking of circles

- The complement $\mathbb{R}^3 - A$ of a single circle A deformation retracts onto $S^1 \vee S^2$.

① $\mathbb{R}^3 - A$ deformation retracts onto the union of S^2 with a diameter.



- points outside S^2 deformation retracts onto S^2
- points inside S^2 and not in A can be pushed away from A toward S^2 or the diameter.

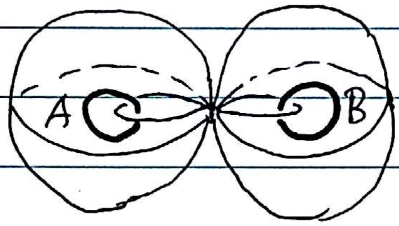
② Then we move the 2 endpoints of the diameter toward each other along the equator until they coincide.

$\pi_1(\mathbb{R}^3 - A) \approx \pi_1(S^1 \vee S^2) \approx \mathbb{Z}$

$\because \pi_1(S^2) = 0.$

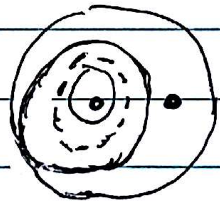
- The complement $\mathbb{R}^3 - (A \cup B)$ of two unlinked circles A and B deformation retracts onto $S^1 \vee S^1 \vee S^2 \vee S^2$.

$\pi_1(\mathbb{R}^3 - (A \cup B)) \approx \mathbb{Z} * \mathbb{Z}.$



- If A and B are linked, then $\mathbb{R}^3 - (A \cup B)$ deformation retracts onto $S^2 \vee (S^1 \times S^1)$

$\pi_1(\mathbb{R}^3 - (A \cup B)) \approx \pi_1(S^1 \times S^1) \approx \mathbb{Z} \times \mathbb{Z}.$



cross section.