Notes for Geometry
Conic Sections-Exercises

The following problems are taken from *Geometry, by David A. Brannan, Matthew F. Esplen and Jeremy J. Gray, 2nd edition*

1. This question concerns the parabola E with equation $y^2 = 2x$ and parametric equations $x = \frac{1}{2}t^2, y = t(t \in \mathbb{R})$.
   (a) Write down the focus, vertex, axis and directrix of E.
   (b) Determine the equation of the chord that joins distinct points P and Q on E with parameters t_1 and t_2, respectively.
   (c) Determine the condition on t_1 and t_2 such that the chord PQ passes through the focus of E.

2. This question concerns the parabola E with equation $y^2 = x$ and parametric equations $x = t^2, y = t(t \in \mathbb{R})$.
   (a) Write down the focus, vertex, axis and directrix of E.
   (b) Determine the equation of the chord that joins the distinct points P and Q on E with parameters t_1 and t_2, respectively.
   (c) Determine the condition on t_1 and t_2 (and so on P and Q) that the focus of E is the midpoint of the chord PQ.

3. Let PQ be an arbitrary chord of the ellipse with equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
   Let M be the midpoint of PQ. Prove that the following expression is independent of the choice of P and Q:

   gradient (i.e. slope) of $OM \times$ gradient of PQ.

4. Let P be an arbitrary point on the ellipse with equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
   and focus $F(ae, 0)$. Let M be the midpoint of FP. Prove that M lies on an ellipse whose center is midway between the origin and F.

5. Let P be a point $(\sec t, \frac{1}{\sqrt{2}} \tan t)$, where $t \in (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$, on the parabola E with equation $x^2 - 2y^2 = 1$.

(a) Determine the foci F and F' of E.

(b) Determine the gradients (i.e. slopes) of FP and $F'P$, when these lines are not parallel to the y-axis.

(c) Determine the point P in the first quadrant on E for which FP is perpendicular to $F'P$.