GEOMETRY FINAL

NAME:______________ ID NO.:____________________ CLASS: __________

1. (10 points) Write down the statement of the Global Gauss-Bonnet theorem.
2. Let T be a torus parametrized by

$$\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \;\; a > b.$$

   Prove that

   (a) (10 points) If a geodesic is tangent to the parallel $u = \frac{\pi}{2}$, then it is entirely contained in the region of T given by $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

   (b) (10 points) A geodesic that intersects the parallel $u = 0$ under an angle ϕ cannot intersect the parallel $u = \pi$ if $\cos\phi > \frac{a-b}{a+b}$.
3. (10 points) Determine whether a (smooth) closed geodesic can bound a simply connected region on a surface when the Gaussian curvature K is nonpositive. If it can, give an example. If it cannot, explain your reason.
4. (10 points) Consider a surface with $K > 0$ that is topologically a cylinder. Prove that there cannot be two disjoint simple closed geodesics both going around the neck of the surface.
5. (20 points) Calculate the Christoffel symbols for the surface of revolution:

$$\mathbf{x}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

   with $f'(u)^2 + g'(u)^2 = 1$.
6. Consider the portion of the paraboloid M parametrized by

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, u^2), 0 \le u \le r, 0 \le v \le 2\pi.$$

   (a) (15 points) Calculate the geodesic curvature of the boundary circle.

   (b) (15 points) Calculate the Gaussian curvature K.

You might need the following formulas.

- When $F = 0$, we have

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

- When $F = 0$, $e_1 = \frac{\mathbf{x}_u}{\sqrt{E}}$ and $e_2 = \frac{\mathbf{x}_v}{\sqrt{G}}$, we have

$$\phi_{12} = \frac{1}{2\sqrt{EG}}\left(-E_v u' + G_u v'\right).$$