GEOMETRY MIDTERM 1

NAME:_____ ID NO.:_____ CLASS: _____

1. (10 points) Determine the image of the line $3x - y + 1 = 0$ under the affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix} \quad (\mathbf{x} \in \mathbb{R}^2).$$

Solution. $5x + y + 4 = 0$. □

2. (10 points) Find the image of the Line $x + 2y - z = 0$ under the projective transformation t defined by

$$t : [x, y, z] \mapsto [2x + y, -x + z, y + z].$$

Solution. $2x + 5y - 4z = 0$. □

3. (15 points) Let $A = [1, 2, 5]$, $B = [1, 0, 3]$ and $C = [2, -5, 1]$ be three Points in $\mathbb{RP}^2$ in homogeneous coordinates.

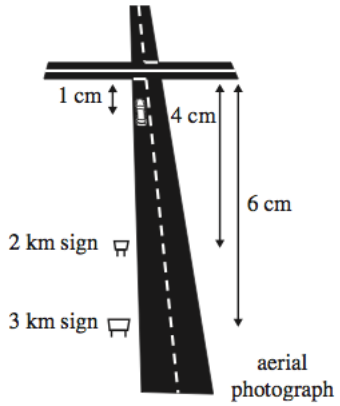
   (a) Show that A, B and C are collinear.

   Proof. Since $\det \begin{pmatrix} 1 & 2 & 5 \\ 1 & 0 & 3 \\ 2 & -5 & 1 \end{pmatrix} = 0$. A, B, C are collinear. □

   (b) Find the Point $D = [a, b, c]$ on the Line through the Points A and B such that the cross-ratio $(ABCD) = 2$.

   Solution. $[1, 20, 23]$. □

4. (15 points) An aerial camera photographs a car traveling along a straight road on flat ground towards a junction. Before the junction there are two warning signs, at distances of 2 km and 3 km from the junction. On the film the signs are 4 cm and 6 cm from the junction, and the car is 1 cm from the junction. How far is the car from the junction on the ground?



Solution. 1/2 km. □

5. (15 points)

   (a) Use the determinant of a matrix to classify the non-degenerate conic

   $$2x^2 + xy - y^2 + 4x - 3y + 3 = 0$$

   in $\mathbb{R}^2$.

   Solution. hyperbola □

   (b) Find the equation for the projective figure in $\mathbb{RP}^2$ which corresponds to the conic $\{(x, y, z) : 2x^2 + xy - y^2 + 4x - 3y + 3 = 0, z = 1\}$ in the standard embedding plane.

   Solution. $2x^2 + xy - y^2 + 4xz - 3yz + 3z^2 = 0$. □

   (c) Which ideal Points should be associated with this projective figure?

   Solution. $[1, 2, 0], [1, -1, 0]$. □

6. (20 points) Let E be the conic in $\mathbb{R}^2$ with the equation

   $$x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0.$$

   Use the methods of linear algebra to answer the following questions.

   (a) To classify the conic E.

   Solution. hyperbola. □

   (b) Write the equation in standard form.

Solution. $\frac{(y'-\sqrt{5})^2}{12} - \frac{(x')^2}{18} = 1$. □

(c) Determine its center/vertex and axis.

Solution. center: $(1, 2)$, major axis: $2x-y = 0$, minor axis: $x+2y = 5$. □

7. (15 points) Determine the affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ which maps the lines $x = 0$, $x - y = 0$ and $y = 1$ to the lines $3x - 2y - 3 = 0$, $x - 1 = 0$ and $4x - y - 9 = 0$, respectively.

Solution. $t(\mathbf{x}) = \begin{pmatrix} -2 & 2 \\ -8 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. □