NAME:________________ ID NO.:________________________ CLASS: ___________

1. (a) (10 points) Prove that for any regular parametrized curve α, we have

$$k = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3}.$$

Proof. Proposition 2.2, page 14. □

(b) (10 points) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be the space curve

$$\alpha(t) = (\cos t, \sin t, t)\,.$$

Calculate the curvature k.

Solution. $k = \frac{1}{2}$. □

2. (20 points) Calculate the Frenet apparatus $(\mathbf{T}, k, \mathbf{N}, \mathbf{B}, \tau)$ of the following curve:

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right), \ \ s \in (-1, 1).$$

Solution. Exercise 1.2.3(a), page 18 □

3. (15 points) Consider the torus

$$\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u) \ \ (0 < b < a), 0 \le u, v \le 2\pi.$$

(a) Compute the first fundamental form of the torus.

Solution. $E = b^2, F = 0, G = (a + b\cos u)^2$. □

(b) Find the surface area of the torus.

Solution. $\int_0^{2\pi}\int_0^{2\pi}\sqrt{EG - F^2}dudv = 4\pi^2 ab$. □

4. (20 points) Consider the helicoid

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, bv).$$

(a) Find the second fundamental form.

(b) Find the matrix of the shape operator.

(c) Find the principal curvatures and the principal directions.

(d) Find the mean curvature H and the Gaussian curvature K.

Solution. HW6. □

5. (15 points) Find the arclength of the tractrix

$$\beta(t) = (t - \tanh t, \operatorname{sech} t), t \geq 0,$$

starting at $(0, 1)$ and proceeding to an arbitrary point.

Solution. HW 4. □

6. (10 points) Prove or give a counterexample: If M is a surface with Gaussian curvature $K > 0$, then the curvature of any curve $C \subset M$ is everywhere positive. (Remember that, by definition, $k \geq 0$.)

Proof. Let C be the curve $\alpha(t)$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Assume that $k = 0$ at some point p in the curve C. Then, by Meusnier's formula and Euler's formula, we have $II_p(v, v) = k_1 \cos^2 \theta + k_2 \sin^2 \theta = k_n = k \cos \phi = 0$. Since $\cos^2 \theta \geq 0$ and $\sin^2 \theta \geq 0$, we have the Gaussian curvature $K(p) = k_1 k_2 \leq 0$. This contradicts the assumption that $K > 0$. □