GEOMETRY FINAL

NAME:________________ ID NO.:________________________ CLASS: ____________

1. (10 points) Write down the statement of the Global Gauss-Bonnet theorem.

2. Let T be a torus parametrized by

$$\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \ \ a > b.$$

   (a) (10 points) Prove that a geodesic that intersects the parallel $u = 0$ under an angle ϕ cannot intersect the parallel $u = \pi$ if $\cos\phi > \frac{a-b}{a+b}$.

   Proof. Use Clairaut's relation. □

   (b) (10 points) Compute the holonomy around the parallel $u = u_0$ on T.

   Solution. $2\pi \sin u_0$ □

3. (10 points) Determine whether a (smooth) closed geodesic can bound a simply connected region on a surface when the Gaussian curvature $K > 0$. If it can, give an example. If it cannot, explain your reason.

   Solution. Yes, it can. Consider great circles on a sphere. □

4. (12 points) Suppose $F = 0$ and the $u-$ and $v-$ curves are geodesics. Prove that the surface is flat (i.e. $K = 0$).

   Proof. First, prove that E is a function depending only on u and G is a function depending only on v. Then use the formula for K below to show that $K = 0$. □

5. (18 points) Calculate the Christoffel symbols for a helicoid:

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, v).$$

   Solution. $\Gamma^u_{uu} = \Gamma^v_{uu} = \Gamma^u_{uv} = \Gamma^v_{vv} = 0$, $\Gamma^v_{uv} = \frac{u}{u^2+1}$, $\Gamma^u_{vv} = -u$. □

6. Consider the portion of the paraboloid M parametrized by

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, u^2), 0 \le u \le r, 0 \le v \le 2\pi.$$

   (a) (15 points) Calculate the geodesic curvature of the boundary circle.

   Solution. $\frac{1}{r\sqrt{4r^2+1}}$ □

(b) (15 points) Calculate the Gaussian curvature K.

Solution. $\frac{4}{(4u^2+1)^2}$ □

You might need the following formulas.

- Geodesic equations

$$\begin{aligned} u''(t) + \Gamma^u_{uu} u'(t)^2 + 2\Gamma^u_{uv} u'(t)v'(t) + \Gamma^u_{vv} v'(t)^2 = 0 \\ v''(t) + \Gamma^v_{uu} u'(t)^2 + 2\Gamma^v_{uv} u'(t)v'(t) + \Gamma^v_{vv} v'(t)^2 = 0 \end{aligned}$$

- When $F = 0$, we have

$$K = -\frac{1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right).$$

- When $F = 0$, $e_1 = \frac{\mathbf{x}_u}{\sqrt{E}}$ and $e_2 = \frac{\mathbf{x}_v}{\sqrt{G}}$, we have

$$\phi_{12} = \frac{1}{2\sqrt{EG}} \left(-E_v u' + G_u v' \right).$$