NAME:_______________ ID NO.:______________________ CLASS: __________

1. (a) (10 pts) Find the arclength of the catenary $\alpha(t) = (t, \cosh t)$ for $0 \le t \le b$.

   Solution. $\sinh b$ □

   (b) (10 pts) Reparametrize the catenary by arclength.

   Solution. $\beta(s) = \left(\log\left(s + \sqrt{s^2+1}\right), \sqrt{s^2+1} \right)$ □

2. (20 pts) Calculate the Frenet apparatus $(\mathbf{T}, k, \mathbf{N}, \mathbf{B}, \tau)$ of the following curve:

$$\alpha(t) = \left(e^t \cos t, e^t \sin t, e^t\right).$$

   Solution. $\mathbf{T} = \frac{1}{\sqrt{3}}(\cos t - \sin t, \cos t + \sin t, 1)$, $k = \frac{\sqrt{2}}{3e^t}$, $\mathbf{N} = \frac{1}{\sqrt{2}}(-\cos t - \sin t, \cos t - \sin t, 0)$, $\mathbf{B} = \frac{1}{\sqrt{6}}(\sin t - \cos t, -\cos t - \sin t, 2)$, $\tau = \frac{1}{3e^t}$ □

3. (10 pts) Suppose C is a simple closed plane curve with $0 < \kappa \le c$. Prove that $\text{length}(C) \ge 2\pi/c$.

   Proof. Let $L = \text{length}(C)$. Then by Hopf rotation theorem, we have

$$2\pi = \int_0^L \kappa(s)\, ds \le \int_0^L c\, ds = c\, L,$$

   so $L \ge 2\pi/c$. □

4. Consider the helicoid

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, bv).$$

   (a) (6 pts) Compute the first fundamental form of the helicoid.

   Solution. $E = 1, F = 0, G = u^2 + b^2$ □

   (b) (10 pts) Find the surface area of a portion of the helicoid $(1 < u < 3, 0 \le v \le 2\pi)$.

   Solution. $\pi\left(3\sqrt{b^2+9} + b^2 \ln|\sqrt{b^2+9}+3| - \sqrt{b^2+1} - b^2 \ln|\sqrt{b^2+1}+1|\right)$ □

5. Consider the torus

$$\mathbf{x}(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u) \quad (0 < b < a).$$

(a) (6 pts) Find the first fundamental form.

Solution. $E = b^2, F = 0, G = (a + b\cos u)^2$ □

(b) (4 pts) Find the surface unit normal.

Solution. $(-\cos u \cos v, -\cos u \sin v, -\sin u)$ □

(c) (6 pts) Find the second fundamental form.

Solution. $l = b, m = 0, n = \cos u(a + b\cos u)$ □

(d) (4 pts) Find the matrix of the shape operator.

Solution. $\begin{pmatrix} 1/b & 0 \\ 0 & \cos u/(a + b\cos u) \end{pmatrix}$ □

(e) (4 pts) Find the mean curvature H and the Gaussian curvature K.

Solution. $H = \frac{1}{2}\left(\frac{1}{b} + \frac{\cos u}{a+b\cos u}\right)$, $K = \frac{\cos u}{b(a+b\cos u)}$ □

6. (10 pts) Apply Meusnier's Formula to prove that the curvature of any curve lying on a sphere of radius a satisfies $\kappa \geq 1/a$.

Proof. By Example 2 of P.46, $S_p = -\frac{1}{a}\operatorname{Id}$. Hence, for any unit vector v and any p on the sphere, $II_p(v,v) = S_p(v) \cdot v = -\frac{1}{a}$. Let $\alpha(s)$ be a unit-speed curve with curvature k on the sphere such that $\alpha(0) = p$ and $\alpha'(0) = v$. By Meusnier's formula,

$$-\frac{1}{a} = II_p(v,v) = k_n = k\cos\phi,$$

where ϕ is the angle between the surface unit normal n and the principal unit normal N of the curve. Therefore, $k \geq |k\cos\phi| = \frac{1}{a}$.

□

2. (*10* points) Evaluate the integrals: $\int \sec x \, dx$ and $\int \sec^3 x \, dx$.

- $\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

  (Let $u = \sec x + \tan x$
  $du = \sec x \tan x + \sec^2 x \, dx$)

  $= \int \frac{1}{u} \, du$

  $= \ln|u| + C$

  $= \ln|\sec x + \tan x| + C$

- $\int \sec^3 x \, dx = \int \underbrace{\sec x}_{f} \cdot \underbrace{\sec^2 x \, dx}_{g'}$

  integration by parts $= \sec x \tan x - \int \sec x \tan x \cdot \tan x \, dx$

  $= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

  $= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$\Rightarrow \quad 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C_1$

$\Rightarrow \quad \int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$