1.3.31.

(a) ($\Rightarrow$) Assume that $v+W$ is a subspace of $V$.

Then $0 = v + (-v) \in v + W$.

$\Rightarrow -v \in W \Rightarrow v \in W$, since $W$ is a subspace of $V$.

($\Leftarrow$). If $v \in W$, then $v + w \in W$ for any $w \in W$ (since $W$ is a subspace of $V$.)

$\Rightarrow v + W \subseteq W$. —— (*)

Let $w \in W$ be any vector in $W$.

$\Rightarrow -v + w \in W$

$\Rightarrow \underbrace{v + (-v + w)}_{= w} \in v + W$

$\Rightarrow W \subseteq v + W$ —— (**)

By (*), (**), we have $W = v + W$.

Hence $v + W$ is a subspace of $V$.

(b). ($\Rightarrow$) Assume $v_1 + W = v_2 + W$.

Let $x \in v_1 + W$.

Then $x = v_1 + w_1$ for some $w_1 \in W$.

Since $v_1 + W = v_2 + W$, $\exists$ some $w_2 \in W$ such that $x = v_1 + w_1 = v_2 + w_2$

$\Rightarrow v_1 - v_2 = w_1 - w_2 \in W$ ($\because W$ is a subspace of $V$.)

$\Leftarrow$) Assume $v_1 - v_2 \in W$.

Let $x \in v_1 + W$. Then $x = v_1 + w_1$ for some $w_1 \in$

Let $y = v_2 + w_2$, where $w_2 = (v_1 - v_2) + w_1$.

Clearly, $y = v_2 + w_2 = v_2 + (v_1 - v_2) + w_1$

$= v_1 + w_1 = x$.

$\Rightarrow x \in v_2 + W$

$\Rightarrow v_1 + W \subseteq v_2 + W$.

Similarly, we have $v_2 + W \subseteq v_1 + W$.

Hence $v_1 + W = v_2 + W$.

(c). Assume that $v_1 + W = v_1' + W$.

Then $v_1 - v_1' \in W$. by (b)

Similarly, assume that $v_2 + W = v_2' + W$.

Then $v_2 - v_2' \in W$

$\because$ W is a subspace of $V$.

$(v_1 - v_1') + (v_2 - v_2') \in W \Rightarrow (v_1 + v_2) - (v_1' + v_2') \in W$

By (b) $\Rightarrow (v_1 + v_2) + W = (v_1' + v_2') + W$.

i.e. $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W$

On the other hand, since $v_1 - v_1' \in W$.

$a(v_1 - v_1') \in W$, $a \in F$

$\Rightarrow av_1 - av_1' \in W$

$\Rightarrow a(v_1 + W) = a(v_1' + W)$.

(d).

(VS1). Let $v_1 + W \in S$ and $v_2 + W \in S$, where $v_1, v_2 \in$

Then $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$.

and $(v_2 + W) + (v_1 + W) = (v_2 + v_1) + W$.

$\because v_1, v_2 \in V \xRightarrow{(VS1)} v_1 + v_2 = v_2 + v_1$.

Hence $(v_1 + v_2) + W = (v_2 + v_1) + W$.

(VS2). Suppose that $v_1 + W$, $v_2 + W$, $v_3 + W \in S$,

where $v_1, v_2, v_3 \in V$.

$$\big((v_1 + W) + (v_2 + W)\big) + (v_3 + W)$$
$$= ((v_1 + v_2) + W) + (v_3 + W) = ((v_1 + v_2) + v_3) + W$$
$$= (v_1 + (v_2 + v_3)) + W = (v_1 + W) + ((v_2 + v_3) + W)$$
$$= (v_1 + W) + \big((v_2 + W) + (v_3 + W)\big).$$

(VS3). $0 + W \in S$

Let $v + W \in S$ for some $v \in V$.

$(0 + W) + (v + W) = (0 + v) + W = v + W$.

(VS4). Let $v + W \in S$, where $v \in V$.

Let $-v + W \in S$, where $-v \in V$. $\because V$ is a V

Thus $v + W + (-v) + W = (v - v) + W = 0 + W$,

where $0 \in S$

(VS5). Let $v+W \in S$ for some $v \in V$.

By (c), $1(v+W) = 1v + W = v + W$.

(VS6). Let $a, b \in F$ and suppose that $v+W \in S$ for some $v \in V$.

Then $(ab)(v+W) = (ab)v + W = a(bv) + W$
$= a(bv + W)$.

(VS7). Let $v_1 + W \in S$ and $v_2 + W \in S$, where $v_1, v_2 \in$

Let $a \in F$.

Then $a((v_1 + W) + (v_2 + W)) = (a(v_1 + v_2)) + W$
$= (av_1 + av_2) + W = (av_1 + W) + (av_2 + W)$
$= a(v_1 + W) + a(v_2 + W)$

(VS8). Let $v + W \in S$, where $v \in V$.

Let $a, b \in F$.

Then $(a+b)(v+W) = ((a+b)v) + W$
$= (av + bv) + W = (av + W) + (bv + W)$
$= a(v + W) + b(v + W)$.

Hence $S$ is a vector space.