NAME:____________ ID NO.:_____________________ CLASS: __________

Problem 1: Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.
Hint: Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for W_1. Since W_1 is a subspace of V. By Replacement Theorem, we can extend β to a basis for V, say $\alpha = \{u_1, \cdots, u_n, u_{n+1}, \cdots, u_m\}$. Let $W_2 = span(\{u_{n+1}, \cdots, u_m\})$. Finish the proof by showing the following two steps.

(1) (4 points) Show that $V = W_1 + W_2$.

Proof. If $v \in V$, then

$$v = \sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{n} a_i u_i + \sum_{i=n+1}^{m} a_i u_i \in W_1 + W_2,$$

for some scalars $a_i, i = 1, \cdots, m$.
This implies that $V \subseteq W_1 + W_2$. But by the definition of $W_1 + W_2$, we also know that $W_1 + W_2 \subseteq V$. Hence $V = W_1 + W_2$.

□

(2) (4 points) Show that $W_1 \cap W_2 = \{0\}$.

Proof. Let $u \in W_1 \cap W_2$. Then $u = \sum_{i=1}^{n} b_i u_i = \sum_{i=n+1}^{m} c_i u_i$, for some scalars $b_1, \cdots, b_n, c_{n+1}, \cdots, c_m$. Then we have

$$\sum_{i=1}^{n} b_i u_i + \sum_{i=n+1}^{m} (-c_i) u_i = 0.$$

But α is linearly independent, since α is a basis. Hence $b_1 = \cdots = b_n = c_{n+1} = \cdots = c_m = 0$. This implies that $u = 0$. That is $W_1 \cap W_2 = \{0\}$. □

Problem 2: Suppose that $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is the linear map defined by

$$T(A) = BA - A^t, \quad \text{where } B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(1) (4 points) Find bases for range and kernel of T.

Solution.

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{is a basis for range of } T$$

and

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{is a basis for kernel of } T.$$

$\square$

(2) (2 points) Find rank and nullity of T.

Solution. rank(T)=3,   nullity(T)=1. $\square$

(3) (3 points) Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be an ordered basis of $M_{2\times 2}(\mathbb{R})$. Find the matrix representation $[T]_\alpha$ of T with respect to α.

Solution.

$$[T]_\alpha = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\square$

(4) (2 points) Compute the determinant of the 4×4 matrix you've found in part (3).

Solution.

$$\det([T]_\alpha) = 0.$$

$\square$

Problem 3: Suppose that $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ is the linear map

$$T(p(x)) = x^2 \frac{d^2 p(x)}{dx^2} + x \frac{dp(x)}{dx} + p(x).$$

(1) (2 points) Show that T is a linear map.

Proof. Let $f(x), g(x) \in P_4(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} & T(f(x) + cg(x)) \\ = \; & x^2 \frac{d^2(f(x) + cg(x))}{dx^2} + x \frac{d(f(x)) + cg(c)}{dx} + (f(x) + cg(x)) \\ = \; & \left(x^2 \frac{d^2 f(x)}{dx^2} + x \frac{df(x)}{dx} + f(x) \right) + c \left(x^2 \frac{d^2 g(x)}{dx^2} + x \frac{dg(x)}{dx} + g(x) \right) \\ = \; & T(f(x)) + cT(g(x)). \end{aligned}$$

Hence T is linear. $\square$

(2) (3 points) Is T one-to-one? Verify your assertion.

Solution. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in P_4(\mathbb{R})$. Then $T(f(x)) = \cdots = 17ax^4 + 10bx^3 + 5cx^2 + 2dx + e = 0$ only when $f(x) = 0$, i.e. $N(T) = \{0\}$. Hence T is one-to-one. □

(3) (2 points) Is T onto? Verify your assertion. (Hint: Use part (2).)

Solution. By the dimension theorem and (2), we have $\dim R(T) = \dim P_4(\mathbb{R})$. Hence $R(T) = P_4(\mathbb{R})$, i.e. T is onto. □

Problem 4: Suppose A is a $n \times n$ square matrix with $A^k = 0$ for some positive integer k, and I is the $n \times n$ identity matrix.

(1) (2 points) Show that $\det(A) = 0$.

Proof.

$$0 = \det(0) = \det(A^k) = (\det(A))^k = 0$$

implies

$$\det(A) = 0.$$

□

(2) (3 points) Show that $I + A$ is invertible by finding its inverse $(I + A)^{-1}$.

Proof. Since

$$\begin{aligned} & (I + A)\left(I - A + A^2 + \cdots + (-1^{k-1})A^{k-1}\right) \\ = \;& \left(I - A + A^2 + \cdots + (-1^{k-1})A^{k-1}\right)(I + A) \\ = \;& I + (-1)^{k-1}A^k = I. \end{aligned}$$

Hence $I + A$ is invertible and

$$(I + A)^{-1} = I - A + A^2 + \cdots + (-1^{k-1})A^{k-1}.$$

□

(3) (3 points) Suppose that x is a $n \times 1$ matrix such that $A^{k-1}x \neq 0$. Show that $\left\{x, Ax, \cdots, A^{k-1}x\right\}$ is linearly independent.

Proof. Let

$$a_0 x + a_1 Ax + \cdots + a_{k-1}A^{k-1}x = 0,$$

where $a_0, a_1, \cdots, a_{k-1}$ are scalars. Multiplying the equality by A^{k-1} from both sides implies that

$$A^{k-1}\left(a_0 x + a_1 Ax + \cdots + a_{k-1}A^{k-1}x\right) = a_0 A^{k-1}x = A^{k-1}0 = 0,$$

where we have used the assumptions that $A^k = 0$ and $A^{k-1}x \neq 0$. Hence $a_0 = 0$. Similarly,

$$A^{k-2}\left(a_1 Ax + \cdots + a_{k-1}A^{k-1}x\right) = a_1 A^{k-1}x = 0$$

implies that $a_1 = 0$. Continuing this process, we have $a_0 = a_1 = \cdots = a_{k-1} = 0$. Hence we have shown that $\left\{x, Ax, \cdots, A^{k-1}x\right\}$ is linearly independent. □

Problem 5: Let $T : P_n(F) \to F^{n+1}$ be the linear transformation defined by $T(f) = (f(c_0), f(c_1), \cdots, f(c_n))$, where $c_0, c_1, \cdots, c_n$ are distinct scalars in an infinite field F. Let β be the standard ordered basis for $P_n(F)$ and γ be the standard ordered basis for F^{n+1}.

(1) (3 points) Show that $M = [T]_\beta^\gamma$ has the form

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

A matrix with this form is called a **Vandermonde matrix**.

Proof.

$$T(1) = (1, 1, \cdots, 1), T(x) = (c_0, c_1, \cdots, c_n), \cdots, T(x^n) = (c_0^n, c_1^n, \cdots, c_n^n)$$

implies that

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

□

(2) (5 points) Prove that

$$\det(M) = \Pi_{0 \leq i < j \leq n}(c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \leq i < j \leq n$.

Proof.

$$\begin{aligned}
f(c_0, c_1, \cdots, c_n) &= \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^n - c_0 c_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n - c_0 & c_n^2 - c_0 c_n & \cdots & c_n^n - c_0 c_n^{n-1} \end{pmatrix} \\
&= \det \begin{pmatrix} c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^n - c_0 c_1^{n-1} & \\ c_2 - c_0 & c_2^2 - c_0 c_2 & \cdots & c_2^n - c_0 c_2^{n-1} & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n - c_0 & c_n^2 - c_0 c_n & \cdots & c_n^n - c_0 c_n^{n-1} & \end{pmatrix} \\
&= (c_1 - c_0)(c_2 - c_0) \cdots (c_n - c_0) \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & c_n & \cdots & c_n^{n-1} \end{pmatrix} \\
&= (c_1 - c_0)(c_2 - c_0) \cdots (c_n - c_0) f(c_1, c_2, \cdots, c_n).
\end{aligned}$$

By induction, $\det(A) = \Pi_{0 \le i < j \le n}(c_j - c_i)$. □

Problem 6: (4 points) Let $A \in M_{n \times n}(F)$ such that

$$A = \begin{pmatrix} 1 + x_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & 1 + x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & 1 + x_n \end{pmatrix}.$$

Compute $\det(A)$.

Solution.

$$\det(A) = 1 + \sum_{i=1}^{n} x_i.$$

□