Problem 1: Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c=0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c\neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer. (5 points)

Solution. No! If $c, d \in \mathbb{R}, c + d \neq 0, c \neq 0, d \neq 0$, then

$$(c + d)(a_1, a_2) = ((c + d)a_1, \frac{a_2}{c + d})$$

usually is not equal to

$$c(a_1, a_2) + d(a_1, a_2) = (ca_1 + da_1, \frac{a_1}{c} + \frac{a_2}{d}).$$

(VS8) does not hold. □

Problem 2: Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. (9 points)

Proof. ($\Leftarrow$) that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$ or W_2.
Since W_1 and W_2 are subspaces V,
we have $W_1 \cup W_2$ is also a subspace of V

($\Rightarrow$) Suppose that $W_1 \cup W_2$ is a subspace of V.
Also suppose that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, then there exist $u, v \in V$ such that $u \in W_1 \backslash W_2, v \in W_2 \backslash W_1$.
$\Rightarrow u, v \in W_1 \cup W_2 \Rightarrow u + v \in W_1 \cup W_2$.
If $u + v \in W_1$, then $(-u) + (u + v) \in W_1 \Rightarrow v \in W_1 \rightarrow\leftarrow$
If $u + v \in W_2$, then $(u + v) + (-v) \in W_2 \Rightarrow u \in W_2 \rightarrow\leftarrow$
Hence $W_1 \subseteq W_2$ or $W_2 \subset W_1$. □

Problem 3: Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (9 points)

Proof. Let $u \in \text{span}(S_1 \cup S_2)$, then $u = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j w_j$, for some scalars $a_i, i = 1, \cdots, m, b_j, j = 1, \cdots, n$, where $v_i, i = 1, \cdots, m$, are in S_1 and $w_j, j = 1, \cdots, n$, are in S_2. Since $\sum_{i=1}^{m} a_i v_i$ is in $\text{span}(S_1)$ and $\sum_{j=1}^{n} b_j w_i$ is in $\text{span}(S_2)$, we have $u \in \text{span}(S_1) + \text{span}(S_2)$. Hence $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$.

Now let $v = x + y \in \text{span}(S_1) + \text{span}(S_2)$, where $x \in \text{span}(S_1)$ and $y \in \text{span}(S_2)$. We can write $x = \sum_{i=1}^{m} a_i v_i$, for some scalars $a_i, i = 1, \cdots, m$ and $v_i \in S_1, i = 1, \cdots, m$ and $y = \sum_{j=1}^{n} b_j w_j$, for some scalars $b_j, j = 1, \cdots, n$ and $w_i \in S_2, j = 1, \cdots, n$. Then we can see that $v = x + y = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j w_j$ is in $\text{span}(S_1 \cup S_2)$, since $v_i, i = 1, \cdots, m, w_j, j = 1, \cdots, n$ are in $S_1 \cup S_2$. Hence $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. Therefore $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. □

Problem 4: Prove that a set S is linear dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \cdots, u_n$ in S such that v is a linear combination of $u_1, u_2, \cdots, u_n$. (9 points)

Proof. ($\Rightarrow$) If S is linearly dependent and $S \neq \{0\}$, then there exist distinct vectors $u_0, u_1, \cdots, u_n \in S$ such that

$$a_0 u_0 + a_1 u_1 + \cdots + a_n u_n = 0$$

with at least one of the scalars $a_0, a_1, \cdots, a_n$ is not zero, say $a_0 \neq 0$.
Then we have

$$u_0 = \left(-\frac{a_1}{a_0}\right) u_1 + \left(-\frac{a_2}{a_0}\right) u_2 + \cdots + \left(-\frac{a_n}{a_0}\right) u_n.$$

Hence $v = u_0$ is a linear combination of $u_1, u_2, \cdots, u_n$.

($\Leftarrow$) If $S = \{0\}$, then it's clear that S is linearly dependent.
Assume that there exist distinct vectors $v, u_1, u_2, \cdots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \cdots, u_n$, say

$$v = a_1 u_1 + a_2 u_2 + \cdots a_n u_n,$$

for some scalars $a_1, a_2, \cdots, a_n$.
Then we have

$$0 = (-1)v + a_1 u_1 + a_2 u_2 + \cdots a_n u_n.$$

Hence S is linearly dependent. □

Problem 5: Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$. (9 points)

Proof. Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for W_1. Since W_1 is a subspace of V. By Replacement Theorem, we can extend β to a basis for V, say $\alpha = \{u_1, \cdots, u_n, u_{n+1}, \cdots, u_m\}$. Let $W_2 = \text{span}(\{u_{n+1}, \cdots, u_m\})$.
Claim that $V = W_1 \oplus W_2$.
1. $\underline{V = W_1 + W_2}$.
If $v \in V$, then

$$v = \sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{n} a_i u_i + \sum_{i=n+1}^{m} a_i u_i \in W_1 + W_2, \quad \text{for some scalars } a_i, i = 1, \cdots, m.$$

This implies that $V \subseteq W_1 + W_2$. But by the definition of $W_1 + W_2$, we also know that $W_1 + W_2 \subseteq V$. Hence $V = W_1 + W_2$.
2. $\underline{W_1 \cap W_2 = \{0\}}$.
Let $u \in W_1 \cap W_2$. Then $u = \sum_{i=1}^{n} b_i u_i = \sum_{i=n+1}^{m} c_i u_i$, for some scalars $b_1, \cdots, b_n, c_{n+1}, \cdots, c_m$. Then we have

$$\sum_{i=1}^{n} b_i u_i + \sum_{i=n+1}^{m} (-c_i) u_i = 0.$$

But α is linearly independent, since α is a basis. Hence $b_1 = \cdots = b_n = c_{n+1} = \cdots = c_m = 0$. This implies that $u = 0$. That is $W_1 \cap W_2 = \{0\}$. By 1 and 2, we have $V = W_1 \oplus W_2$. We are done! □

Problem 6: Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. (9 points)
(Hint: Start with a basis $\{u_1, u_2, \cdots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$ for W_1 and to a basis $\{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_p\}$ for W_2.)

Proof. $\dim(W_1 \cap W_2) \leq \dim(V)$
$\Rightarrow W_1 \cap W_2$ has a finite basis $\beta = \{u_1, u_2, \cdots, u_k\}$.
We can extend β to a basis $\beta_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$ for W_1 and to a basis $\beta_2 = \{u_1, u_2, \cdots, u_k, k_1, k_2, \cdots, k_p\}$ for W_2.
Let $\alpha = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$.
We claim that α is a basis for $W_1 + W_2$.

To prove the claim, we need to check that

1. <u>α is linearly independent.</u>

Let $a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m + c_1w_1 + \cdots + c_pw_p = 0$, for some scalars $a_1, \cdots, a_k, b_1, \cdots, b_m, c_1, \cdots, c_p$.
Then $(-b_1)v_1 + \cdots + (-b_m)v_m = a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots + c_pw_p \in W_1 \cap W_2$.
Since β is a basis for $W_1 \cap W_2$, we have $(-b_1)v_1 + \cdots + (-b_m)v_m = d_1u_1 + \cdots + d_ku_k$ for some scalars $d_1, \cdots, d_k$.
$\Rightarrow d_1u_1 + \cdots + d_ku_k + b_1v_1 + \cdots + b_mv_m = 0$
$\Rightarrow d_1 = \cdots = d_k = b_1 = \cdots = b_m = 0$, since β_1 is a basis for W_1.
$\Rightarrow a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots + c_pw_p = 0$
$\Rightarrow a_1 = \cdots = a_k = c_1 = \cdots = c_p = 0$, since β_2 is a basis for W_2.
Hence α is linearly independent.

2. <u>$W_1 + W_2 = \text{span}(\alpha)$.</u>

Let $u = v + w \in W_1 + W_2$, where $v \in W_1$ and $w \in W_2$, be any vector in $W_1 + W_2$.
Since β_1 is a basis for W_1 and β_2 is a basis for W_2, we can find some scalars $x_1, \cdots, x_k, y_1, \cdots, y_m, z_1, \cdots, z_k,$
such that

$$\begin{aligned} u &= (x_1u_1 + \cdots + x_ku_k + y_1v_1 + \cdots y_mv_m) + (z_1u_1 + \cdots + z_ku_k + t_1w_1 + \cdots + t_pw_p) \\ &= \big((x_1 + z_1)u_1 + \cdots + (x_k + z_k)u_k + y_1v_1 + \cdots y_mv_m + t_1w_1 + \cdots + t_pw_p\big) \end{aligned}$$

That is, $W_1 + W_2 \subseteq \text{span}(\alpha)$.
It is easy to see that $\text{span}(\alpha) \subseteq W_1 + W_2$.
Hence $W_1 + W_2 = \text{span}(\alpha)$.
Therefore, α is a basis for $W_1 + W_2$.

Finally, we have

$$\dim(W_1 + W_2) = k + m + p = (k + m) + (k + p) - k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

□

Bonus Problem 7: Prove that the intersection of three 6-dimensional subspaces in $\mathbb{R}^8$ is not the zero vector space $\{0\}$. (5 points)

Proof. Let U, V and W be three 6-dimensional subspaces of $\mathbb{R}^8$.
Then $U + V$ is a subspace of $\mathbb{R}^8$ and $\dim(U + V) \leq 8$.

By Problem 6, we have

$$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) \geq 6 + 6 - 8 = 4.$$

Similarly, we have $\dim(U \cap W) \geq 4$.
Since $U \cap V \subseteq U$ and $U \cap W \subseteq U$, $U \cap V$ and $U \cap W$ are subspaces of U.
This implies that $(U \cap V) + (U \cap W)$ is a subspace of U.
Hence, $\dim\left((U \cap V) + (U \cap W)\right) \leq \dim(U) = 6$.
By Problem 6 again, we have

$$\begin{aligned}
\dim(U \cap V \cap W) &= \dim\left((U \cap V) \cap (U \cap W)\right) \\
&= \dim(U \cap V) + \dim(U \cap W) - \dim\left((U \cap V) + (U \cap W)\right) \\
&\geq 4 + 4 - 6 = 2.
\end{aligned}$$

Therefore, the intersection of three 6-dimensional subspaces in $\mathbb{R}^8$ has at least dimension two and can not be the zero vector space $\{0\}$. $\square$