NAME:______________ ID NO.:__________________________ CLASS: ____________

Problem 1: Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(1) Prove that if UT is one-to-one, then T is one-to-one. (3 points)

Proof. Suppose that UT is one-to-one, then $N(UT) = \{0\}$. Let $v \in V$ such that $T(v) = 0$, then $U(T(v)) = UT(v) = 0$, i.e. $v \in N(UT) = \{0\}$. Hence $v = 0$. That means $N(T) = \{0\}$, i.e. T is one-to-one. □

(2) Prove that if UT is onto, then U is onto. (3 points)

Proof. Suppose that UT is onto, then for all $z \in Z$, there exists a $v \in V$ such that $UT(v) = z$. This implies that, for all $z \in Z$, there exists a $T(v) \in W$ such that $U(T(v)) = UT(v) = z$. Therefore, U is onto. □

(3) Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. (5 pints)

Proof. Suppose that AB is invertible, then, by Theorem, L_{AB} is invertible. This implies that $L_{AB} = L_A L_B$ is one-to-one and onto. By (1) and (2), L_A is onto and L_B is one-to-one. Since L_A, L_B, L_{AB} are linear maps from F^n to F^n and $\dim F^n = \dim F^n = n$, by Theorem 2.5, we have L_A is one-to-one and L_B is onto. Hence L_A and L_B are invertible. This implies that A and B are invertible. □

Problem 2: Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear. Assume that $\text{rank}(T) = \text{rank}(T^2)$.

(1) Suppose that $\beta = \{v_1, \cdots, v_n\}$ is a basis for $R(T)$. Show that $\{T(v_1), \cdots, T(v_n)\}$ forms a basis for $R(T^2)$. (4 points)

Proof. For any $u \in R(T^2)$, there exists $v \in R(T)$ such that $u = T(v)$. Let $v = \sum_{i=1}^n a_i v_i$, for some scalars $a_1, \cdots, a_n$. Then $u = T(v) = \sum_{i=1}^n a_i T(v_i)$, i.e. $\text{span}(\{T(v_1), \cdots, T(v_n)\}) = R(T^2)$. The fact that $\text{span}(\{T(v_1), \cdots, T(v_n)\}) = R(T^2)$ and $\text{rank}(T) = \text{rank}(T^2) = n$, by assumption, imply that $\{T(v_1), \cdots, T(v_n)\}$ forms a basis for $R(T^2)$. □

(2) Prove that $R(T) \cap N(T) = \{0\}$. (4 points)

Proof. Let $w \in R(T) \cap N(T)$, then $w = \sum_{i=1}^n b_i v_i$, for some scalars $b_1, \cdots, b_n$, and $T(w) = 0$. This implies $\sum_{i=1}^n b_i T(v_i) = 0$. Since $\{T(v_1), \cdots, T(v_n)\}$ is a basis for $R(T^2)$, we have $b_1 = \cdots = b_n = 0$. Hence $w = 0$ and $R(T) \cap N(T) = \{0\}$. □

(3) Deduce that $V = R(T) \oplus N(T)$. (4 points)

Proof. Note that $R(T) + N(T) \subseteq V$, since $R(T)$ and $N(T)$ are subspaces of V. We also have $\dim(R(T) + N(T)) = \dim R(T) + \dim N(T) - \dim(R(T) \cap N(T)) = \dim R(T) + \dim N(T) = \dim V$, where the last equality follows from the Dimension Theorem. Therefore $V = R(T) \oplus N(T)$. □

Problem 3: Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation.

(1) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. (9 points)

Proof. ($\Rightarrow$) Suppose that T is one-to-one. Let S be a linearly independent subset of V. We want to show that $T(S)$ is linearly independent. Suppose that $T(S)$ is linearly dependent. Then there exist $v_1, \cdots, v_n \in S$ and some not all zero scalars $a_1, \cdots, a_n$ such that

$$a_1 T(v_1) + \cdots + a_n T(v_n) = 0.$$

Since T is linear,

$$a_1 T(v_1) + \cdots + a_n T(v_n) = T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

By assumption that T is one-to-one, we also know that $N(T) = \{0\}$. Hence

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

But S is linearly independent and $v_1, \cdots, v_n \in S$, we have $a_1 = \cdots = a_n = 0$.
$\rightarrow\leftarrow$
Since S is arbitrary, T carries linearly independent subsets of V onto linerly independent subsets of W.
($\Leftarrow$) Suppose that T carries linearly independent subsets of V onto linearly independent subsets of W. Assume that $T(x) = 0$. If the set $\{x\}$ is linearly independent, then by assumption we conclude that $\{0\}$ is linearly independent, which is a contradiction. Hence the set $\{x\}$ is linearly dependent. This implies that $x = 0$. That is, $N(T) = \{0\}$. Therefore, T is one-to-one. □

(2) Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W. (8 points)

Proof. ($\Rightarrow$) T is an isomorphism. $\Rightarrow$ T is invertible. $\Rightarrow$ T is injective and surjective. Since T is injective, by 2.1.14 (a), we know that $T(\beta)$ is linearly independent. Since T is surjective, we know that $\text{span}(T(\beta)) = R(T) = W$. Hence $T(\beta)$ is a basis for W.

($\Leftarrow$) Since $T(\beta)$ is a basis, we know $\text{span}(T(\beta)) = R(T) = W$. This implies that T is surjective and $\dim R(T) = \dim W = n$. By assumption, we have $\dim V = \dim W = n$. By Dimension Theorem, we have $n = \dim V = \dim R(T) + \dim N(T)$. This implies that $\dim N(T) = 0$, i.e. $N(T) = \{0\}$. So T is injective. Therefore, we show that T is an isomorphism. $\square$

Problem 4: Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the map defined by

$$T(p(x)) = \frac{d^2p(x)}{dx^2} + 2\frac{dp(x)}{dx},$$

for all $p(x) \in P_3(\mathbb{R})$.

(1) Show that T is a linear transformation. (3 points)

Proof. Let $f(x), g(x) \in P_3(\mathbb{R})$ and $c \in \mathbb{R}$. Then $T(cf(x) + g(x)) = \frac{d^2(cf(x)+g(x))}{dx^2} + 2\frac{d(cf(x)+g(x))}{dx} = c\left(\frac{d^2f(x)}{dx^2} + 2\frac{df(x)}{dx}\right) + \left(\frac{d^2g(x)}{dx^2} + 2\frac{dg(x)}{dx}\right) = cT(f(x)) + T(g(x))$. Hence T is linear. $\square$

(2) Find the matrices $[T]_\alpha$ and $[T]_\beta$ representing T with respect to the ordered bases $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$, respectively. (4 points)

Solution. $[T]_\alpha = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $[T]_\beta = \begin{pmatrix} 0 & 2 & 0 & -6 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. $\square$

(3) Find the inverse matrix A^{-1} of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Hint: Notice that the matrix $A = [I]_\beta^\alpha$ is the change of coordinate matrix that change β-coordinates into α-coordinates. (4 points)

Solution. $A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $\square$