NAME:____________ ID NO.:____________________ CLASS: __________

Problem 1: (4 points) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 4 \\ 2 & 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 5 \end{pmatrix}$. Try to find an elementary matrix such that $B = AE$.

Solution. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

□

Problem 2: (8 points) Express the invertible matrix $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ as a product of elementary matrices.

Solution.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

□

Problem 3: (8 points) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by

$$T(f(x)) = f(x) + f'(x) + f''(x).$$

Determine whether T is invertible, and compute T^{-1} if it exists.

Solution. See the solution for Example 7 of Sec.3.2 in the textbook.

□

Problem 4: Let $T, U : V \to W$ be linear transformations.

(1) (4 points) Prove that

$$R(T + U) \subseteq R(T) + R(U).$$

(2) (4 points) Prove that if W is finite-dimensional, then

$$\text{rank}(T + U) \leq \text{rank}(T) + \text{rank}(U).$$

(3) (4 points) Deduce from (2) that

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$$

for any $m \times n$ matrices A and B.

Proof of (1). For any $v \in R(T+U)$, we can write it as $v = (T+U)(w) = T(w) + U(w) \in R(T) + R(U)$ for some $w \in V$. Hence $R(T+U) \subseteq R(T) + R(U)$. □

Proof of (2).

$$\begin{aligned} \operatorname{rank}(T+U) &= \dim(R(T+U)) \\ &\overset{(a)}{\leq} \dim(R(T)+R(U)) \\ &\overset{1.6.29(a)}{=} \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U)) \\ &\leq \dim(R(T)) + \dim(R(U)) \\ &= \operatorname{rank}(T) + \operatorname{rank}(U). \end{aligned}$$

□

Proof of (3).

$$\operatorname{rank}(A+B) = \operatorname{rank}(L_{A+B}) = \operatorname{rank}(L_A + L_B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

□

Problem 5: (8 points) Let the reduced row echelon form of A be $\begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$.

Determine A if the first, second, and fourth columns of A are $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, respectively.

Solution. $A = \begin{pmatrix} 1 & 0 & 3 & 1 & 7 \\ -1 & -1 & 1 & -2 & -9 \\ 3 & 1 & 5 & 0 & 3 \end{pmatrix}$. □

Problem 6: (8 points) Let W be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of the symmetric 2×2 matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates W. Find a subset of S that is a basis for W.

Proof.

$$\left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}$$

forms a basis for W. □