1.3.19:

Proof. ($\Leftarrow$) that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$ or W_2.
Since W_1 and W_2 are subspaces V,
we have $W_1 \cup W_2$ is also a subspace of V

($\Rightarrow$) Suppose that $W_1 \cup W_2$ is a subspace of V.
Also suppose that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, then there exist $u, v \in V$ such that $u \in W_1 - W_2, v \in W_2 - W_1$.
$\Rightarrow u, v \in W_1 \cup W_2 \Rightarrow u + v \in W_1 \cup W_2$.
If $u + v \in W_1$, then $(-u) + (u + v) \in W_1 \Rightarrow v \in W_1 \rightarrow\leftarrow$
If $u + v \in W_2$, then $(u + v) + (-v) \in W_2 \Rightarrow u \in W_2 \rightarrow\leftarrow$
Hence $W_1 \subseteq W_2$ or $W_2 \subset W_1$. □

1.3.23 (a)

Proof. Assume that W_1, W_2 are subspaces of V.
1. We have $0 = 0 + 0 \in W_1 + W_2$.
2. Let $x_1, y_1 \in W_1$, then $x_1 + y_1 \in W_1$.
Similarly, let $x_2, y_2 \in W_2$, then $x_2 + y_2 \in W_2$.
By the definition of $W_1 + W_2$, we have $(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) \in W_1 + W_2$.
3. Let $x_1 \in W_1, x_2 \in W_2$ and c be a scalar, then $c(x_1 + x_2) = cx_1 + cx_2 \in W_1 + W_2$.
Finally, $W_1 = \{x + 0 : x \in W_1, 0 \in W_2\} \subseteq W_1 + W_2$ and, similarly, $W_2 = \{0 + x : 0 \in W_1, x \in W_2\} \subseteq W_1 + W_2$. □

1.3.23 (b)

Proof. Suppose that W is a subspace of V containing both W_1 and W_2. We want to show that $W_1 + W_2 \subseteq W$.
Let $u = x + y \in W_1 + W_2$ for some $x \in W_1, y \in W_2$.
Since $x \in W$ and $y \in W$, then we have $x + y \in W$.
Hence $W_1 + W_2 \subseteq W$.

□

1.3.30:

Proof. ($\Rightarrow$) Suppose that W_1, W_2 are subspaces of V and $V = W_1 \oplus W_2$.
Then, for any vector v in V, if $v = x_1 + x_2 = y_1 + y_2$, where $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, then $x_1 - y_1 = y_2 - x_2 \in W_1 \cap W_2 = \{0\}$.
$\Rightarrow x_{=} y_1, x_2 = y_2$.

($\Leftarrow$) Clearly we have $V = W_1 + W_2$.
If $W_1 \cap W_2$ contains a nonzero vector x, then we have $x = x + 0 \in W_1 + W_2$ and also $x = 0 + x \in W_1 + W_2$ which contradicts to our hypothesis that x can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1, x_2 \in W_2$.

□