1.6.21:

Proof. (⇒) Let V be an infinite dimensional vector space, then $V \neq \{0\}$.
Pick up a nonzero vector v_1 in V.
Then $\text{span}\{v_1\} \neq V$, otherwise $\dim(V) = 1$ which contradicts our assumption.
So there exists a nonzero vector v_2 in V such that $v_2 \notin \text{span}\{v_1\}$.
By Theorem 1.7, $\{v_1, v_2\}$ is linearly independent.
Continuing this process, we obtain an infinite linearly independent subset of V.
(⇐) Assume that V contains an infinite linearly independent subset α.
Suppose that V is finite dimensional, say $\dim V = n$, and β is a basis of V, then $\#(\beta) = n$.
Let γ be an subset of α and $\#(\gamma) = n + 1$.
Then γ is linearly independent.
By Replacement Theorem, we have the contradiction that $n + 1 \geq n$.
Hence, V is infinite dimensional. □

1.6.22:

Proof. We claim that $W_1 v \subseteq W_2$ if and only if $\dim(W_1 \cap W_2) = \dim(W_1)$.
(⇐) If $W_1 \subseteq W_2$, then $W_1 \cap W_2 = W_1$.
Hence $\dim(W_1 \cap W_2) = \dim(W_1)$.
(⇒) Assume that $\dim(W_1 \cap W_2) = \dim(W_1)$.
Since W_1 is a subspace of V, hence $\dim(W_1) < \infty$.
Theorem 1.11 says that $W_1 \cap W_2 = W_1$.
Hence $W_1 \subseteq W_2$. □

1.6.29(a):

Proof. $\dim(W_1 \cap W_2) \leq \dim(V)$
$\Rightarrow W_1 \cap W_2$ has a finite basis $\beta = \{u_1, u_2, \cdots, u_k\}$.
We can extend β to a basis $\beta_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$ for W_1 and to a basis $\beta_2 = \{u_1, u_2, \cdots, u_k, k_1, k_2, \cdots, k_p\}$ for W_2.
Let $\alpha = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$.
We claim that α is a basis for $W_1 + W_2$.
To proof the claim, we need to check that
1. $W_1 + W_2 = \text{span}(\alpha)$.
$\cdots$
2. α is linearly independent.
$\cdots$ □

1.6.34(a):

Proof. Assume that V is a finite-dimensional vector space with $\dim(V) = n$.
Let $\alpha = \{u_1, u_2, \cdots, u_m\}$ be a basis of W_1.
By Replacement Theorem, we can extend α to a basis $\beta = \{u_1, u_2, \cdots, u_m, u_{m+1}, u_{m+2}, \cdots, u_n\}$ of V.
Let $W_2 = \text{span}(\{u_{m+1}, u_{m+2}, \cdots, u_n\})$.
<u>Claim</u>: $V = W_1 \oplus W_2$.
1. Check $V = W_1 + W_2$.
$\cdots$
2. Check $W_1 \cap W_2 = \{0\}$.
$\cdots$ □