2.3.16 (a):

Proof. For any $u \in R(T^2)$, there exists $v \in R(T)$ such that $u = T(v)$. Let $\beta = \{v_1, \cdots, v_n\}$ be a basis for $R(T)$ and $v = \sum_{i=1}^n a_i v_i$, for some scalars $a_1, \cdots, a_n$. Then $u = T(v) = \sum_{i=1}^n a_i T(v_i)$, i.e. $\text{span}(\{T(v_1), \cdots, T(v_n)\}) = R(T^2)$. Since $\text{rank}(T) = \text{rank}(T^2) = n$. This implies that $\{T(v_1), \cdots, T(v_n)\}$ forms a basis for $R(T^2)$. Let $w \in R(T) \cap N(T)$, then $w = \sum_{i=1}^n b_i v_i$, for some scalars $b_1, \cdots, b_n$, and $T(w) = 0$. This implies $\sum_{i=1}^n b_i T(v_i) = 0$. Since $\{T(v_1), \cdots, T(v_n)\}$ is a basis for $R(T^2)$, we have $b_1 = \cdots = b_n = 0$. Hence $w = 0$ and $R(T) \cap N(T) = \{0\}$.
Note that $R(T) + N(T) \subseteq V$, since $R(T)$ and $N(T)$ are subspaces of V. We also have $\dim(R(T) + N(T)) = \dim R(T) + \dim N(T) - \dim(R(T) \cap N(T)) = \dim R(T) + \dim N(T) = \dim V$, where the last equality follows from the Dimension Theorem. Therefore $V = R(T) \oplus N(T)$. □

2.3.16 (b):

Proof. First note that $\text{rank}(T^{i+1}) \leq \text{rank}(T^i)$, since $T^{i+1}(V) = T^i(R(V)) \subseteq T^i(V)$. But $\text{rank}(T^i)$ is an integer and $0 \leq \text{rank}(T^i) \leq \dim V$. So there exists some integer k such that $\text{rank}(T^k) = \text{rank}(T^{k+1})$ and hence $T^k(V) = T^{k+1}(V)$. Hence $T^k(V) = T^i(V)$ for all $i \geq k$. So we have $\text{rank}(T^k) = \text{rank}(T^{2k})$. By (a), we have $V = R(T^k) \oplus N(T^k)$ for some integer k. □

2.3.17:

Proof. Note that for $x = T(x) + (x - T(x))$ for every $x \in V$. By assumption, $T(T(x)) = T(x)$, so $T(x) \in \{y : T(y) = y\}$ and $x - T(x) \in N(T)$. So $V = \{y : T(y) = y\} + N(T)$.
If $y \in \{y : T(y) = y\} \cap N(T)$, then $x = T(x) = 0$, i.e. $\{y : T(y) = y\} \cap N(T) = \{0\}$. Hence $V = \{y : T(y) = y\} \oplus N(T)$.
(It is enough for you to show that $V = \{y : T(y) = y\} \oplus N(T)$. If fact, T is a projection on W_1 along W_2 for some subspaces W_1 and W_2 of V such that $V = W_1 \oplus W_2$.)

□