Problem 1: Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (2a_1 + b_1, a_2 + 2b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.
Solution. No!
We have

$$(a_1, a_2) + (b_1, b_2) = (2a_1 + b_1, a_2 + 2b_2).$$

But

$$(b_1, b_2) + (a_1, a_2) = (2b_1 + a_1, b_2 + 2a_2).$$

So, (VS1) fails to hold.
Hence V is not a vector space over $\mathbb{R}$ with these operations.

Problem 2: Let W_1, W_2 be subspaces of a vector space V. State whether the following is also a subspace of V. Prove or give a counterexample.

(1) $W_1 \cap W_2$

Proof. 1. Since W_1 and W_2 are subspaces of V, $0 \in W_1$ and $0 \in W_2$. This implies $0 \in W_1 \cap W_2$.
2. Let $x, y \in W_1 \cap W_2$. Then $x, y \in W_1$ and $x + y \in W_1$, since W_1 is a subspace of V. Similarly, $x, y \in W_2$ and $x+y \in W_2$, since W_2 is a subspace of V. This implies $x + y \in W_1 \cap W_2$.
3. Let $x \in W_1 \cap W_2$ and c be a scalar. Then $x \in W_1$ and $cx \in W_1$, since W_1 is a subspace of V. Similarly, $x \in W_2$ and $cx \in W_2$, since W_2 is a subspace of V. This implies $cx \in W_1 \cap W_2$.
Hence $W_1 \cap W_2$ is a subspace of V. □

(2) $W_1 \cup W_2$
Solution. No, $W_1 \cup W_2$ may not be a subspace of V.
For example, let $V = \mathbb{R}^2, W_1 = x-$axis and $W_2 = y-$axis, then $W_1 \cup W_2$ is not a subspace of V. Since $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

(3) $(V - W_1) \cap W_2$.
Solution. No, $(V - W_1) \cap W_2$ may not be a subspace of V.
For example, let $V = \mathbb{R}^2, W_1 = x -$ axis and $W_2 = y -$ axis, then $(V - W_1) \cap W_2 = y -$ axis$\setminus\{(0, 0)\}$. Since there is no zero vector, it is not s subspace of $\mathbb{R}^2$.