NAME:_____________ ID NO.:______________________ CLASS: __________

Problem 1: Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$. (4 points)

Proof. ($\Rightarrow$) It is clear that $W \subseteq \text{span}(W)$.
We need to show that if W is a subspace of V, then $\text{span}(W) \subseteq W$.
For any $u \in \text{span}(W)$,

$$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for some $v_1, v_2, \cdots, v_n \in W$ and some scalars $a_1, a_2, \cdots, a_n$.
Since W is a subspace of V and $v_1, v_2, \cdots, v_n \in W$,

$$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n \in W.$$

So, $\text{span}(W) \subseteq W$.
Hence, if W is a subspace of V, then W $= W$.

($\Leftarrow$) By Theorem 1.5, we have that $\text{span}(W) = W$ is a subspace of V. □

Problem 2: Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \cdots, u_n$ in S such that v is a linear combination of $u_1, u_2, \cdots, u_n$. (5 points)

Proof. ($\Rightarrow$) If S is linearly dependent and $S \neq \{0\}$, then there exist distinct vectors $u_0, u_1, \cdots, u_n \in S$ such that

$$a_0u_0 + a_1u_1 + \cdots + a_nu_n = 0$$

with at least one of the scalars $a_0, a_1, \cdots, a_n$ is not zero, say $a_0 \neq 0$.
Then we have

$$u_0 = \left(-\frac{a_1}{a_0}\right)u_1 + \left(-\frac{a_2}{a_0}\right)u_2 + \cdots + \left(-\frac{a_n}{a_0}\right)u_n.$$

Hence $v = u_0$ is a linear combination of $u_1, u_2, \cdots, u_n$.

($\Leftarrow$) If $S = \{0\}$, then it's clear that S is linearly dependent.
Assume that there exist distinct vectors $v, u_1, u_2, \cdots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \cdots, u_n$, say

$$v = a_1u_1 + a_2u_2 + \cdots a_nu_n,$$

for some scalars $a_1, a_2, \cdots, a_n$.
Then we have

$$0 = (-1)v + a_1u_1 + a_2u_2 + \cdots a_nu_n.$$

Hence S is linearly dependent. □

Problem 3:

(1) Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are not equal. (3 points)

Solution of (1). For example, let $S_1 = \{(1,0)\}$ and $S_2 = \{(2,0)\}$, then

$$\text{span}(S_1 \cap S_2) = \text{span}(\phi) = \{(0,0)\}$$

and

$$\text{span}(S_1) \cap \text{span}(S_2) = x - \text{axis}.$$

(2) Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^t$ and $g(t) = e^{2t}$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. (3 points)

Solution of (2). Let

$$ae^t + be^{2t} = 0,$$

where $a, b \in \mathbb{R}$.

Differentiate the equation with respect to t on both sides, we obtain

$$ae^t + 2be^{2t} = 0.$$

By solving the system of the equations, we obtain $a = b = 0$.

Hence f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.