NAME:_____________ ID NO.:_______________________ CLASS: ___________

Problem 1: The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathrm{M}_{n\times n}(F)$. Find a basis for W. What is the dimension of W? (5 points)
Solution.

$\beta = \{E_{ij}, i \neq j, -E_{11} + E_{kk}, k = 2, \cdots, n\}$ is a basis for W.

$\dim W = n^2 - 1$.

Problem 2: Let

$$V = M_{2\times 2}(F), \quad W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\},$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$. (Hint: $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.) (10 points)
Solution.

Show that $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for W_1.
Then $\mathrm{span}(\beta) = W_1$ is a subspace of V by Theorem 1.5.

Show that $\beta' = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for W_2.
Then $\mathrm{span}(\beta') = W_2$ is a subspace of V by Theorem 1.5.

Show that $\gamma = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ is a basis for $W_1 \cap W_2$.

$\dim W_1 = 3$, $\dim W_2 = 2, \dim W_1 \cap W_2 = 1. \dim W_1 + W_2 = 4.$