NAME:____________ ID NO.:______________________ CLASS: __________

Problem 1: Let V and W be vector spaces and $T : V \to W$ be linear. Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. (9 points)

Proof. ($\Rightarrow$) Suppose that T is one-to-one. Let S be a linearly independent subset of V. We want to show that $T(S)$ is linearly independent. Suppose that $T(S)$ is linearly dependent. Then there exist $v_1, \cdots, v_n \in S$ and some not all zero scalars $a_1, \cdots, a_n$ such that

$$a_1T(v_1) + \cdots + a_nT(v_n) = 0.$$

Since T is linear,

$$a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) = 0.$$

By assumption that T is one-to-one, we also know that $N(T) = \{0\}$. Hence

$$a_1v_1 + \cdots + a_nv_n = 0.$$

But S is linearly independent and $v_1, \cdots, v_n \in S$, we have $a_1 = \cdots = a_n = 0$. $\rightarrow\leftarrow$

Since S is arbitrary, T carries linearly independent subsets of V onto linerly independent subsets of W.

($\Leftarrow$) Suppose that T carries linearly independent subsets of V onto linearly independent subsets of W. Assume that $T(x) = 0$. If the set $\{x\}$ is linearly independent, then by assumption we conclude that $\{0\}$ is linearly independent, which is a contradiction. Hence the set $\{x\}$ is linearly dependent. This implies that $x = 0$. That is, $N(T) = \{0\}$. Therefore, T is one-to-one. □

Problem 2: Let $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be a linear transformation defined by $T(A) = \operatorname{tr}(A)$. Recall that $\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$.

(1) Find a basis for $N(T)$. (6 points)
(2) Find a basis for $R(T)$. (3 points)

Solution. (1) $\beta = \{E_{ij}, i \neq j, -E_{11} + E_{kk}, k = 2, \cdots, n\}$ is a basis for $N(T)$.
(2) $\{1\}$ is a basis for $R(T)$.

□