Name:____________ Id No.:_____________________ Class: __________

Problem 1: Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$. (8 points)

Proof. Suppose that $\{T, U\}$ is linearly dependent. Then there exists some nonzero scalar $cT = U$. Since T is a nonzero transformation from V to W, there exists some vector $u \in V$ and some nonzero vector $v \in W$ such that $T(u) = v \neq 0$. Then $U(u) = cv$. But we also have $v = \frac{1}{c}(cv) = \frac{1}{c}U(u) = U(\frac{1}{c}u) \in R(U)$. This implies that $0 \neq v \in R(T) \cap R(U)$ which contradicts our assumption. Hence $\{T, U\}$ is linearly independent. □

Problem 2: Let $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$U(a + bx + cx^2) = (a - b, c, a + b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. Let $h(x) = 1 - 2x + 4x^2$. Compute $[U]_\beta^\gamma$, $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then verify that $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta$. (10 points)

Solution. $[U]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $[h(x)]_\beta = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, $[U(h(x))]_\gamma = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$. □