NAME:____________ ID NO.:____________________ CLASS: __________

Problem 1:

(1) (3 points) Let the rows of $A \in M_{n\times n}(F)$ be $a_1, a_2, \cdots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \cdots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

Solution. $\det(B) = (-1)^{\frac{n(n-1)}{2}} \det(A)$. □

(2) (3 points) A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible.

Proof. $\det(-M) = (-1)^n \det(M)$ and $\det(M^t) = \det(M)$. Hence if n is odd, then $\det(M) = 0$. This implies that M is not invertible. □

(3) (3 points) Evaluate the determinant of $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ by cofactor expansion along the second row.

Solution. -12 □

Problem 2: (6 points) Let $A \in M_{n+1\times n+1}(F)$ be **a Vandermonde matrix** such that

$$A = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

Show that $\det(A) = \Pi_{0\le i<j\le n}(c_j - c_i)$.

Proof.

$$\begin{aligned}
f(c_0, c_1, \cdots, c_n) &= \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^n - c_0 c_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n - c_0 & c_n^2 - c_0 c_n & \cdots & c_n^n - c_0 c_n^{n-1} \end{pmatrix} \\
&= \det \begin{pmatrix} c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^n - c_0 c_1^{n-1} \\ c_2 - c_0 & c_2^2 - c_0 c_2 & \cdots & c_2^n - c_0 c_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n - c_0 & c_n^2 - c_0 c_n & \cdots & c_n^n - c_0 c_n^{n-1} \end{pmatrix} \\
&= (c_1 - c_0)(c_2 - c_0)\cdots(c_n - c_0) \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & c_n & \cdots & c_n^{n-1} \end{pmatrix} \\
&= (c_1 - c_0)(c_2 - c_0)\cdots(c_n - c_0) f(c_1, c_2, \cdots, c_n).
\end{aligned}$$

By induction, $\det(A) = \Pi_{0\le i<j\le n}(c_j - c_i)$. □