Linear Algebra II　　　　Final Exam

Name:____________ Id No.:_____________________ Class: __________

Problem 1: Let $A = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

(1) (3 points) Find the minimal polynomial of A.

Solution. $(t-2)^2(t-3)$ □

(2) (3 points) Find a Jordan canonical form J of A.

Solution. $J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. □

(3) (5 points) Find a matrix Q such that $Q^{-1}AQ = J$.

Solution. $Q = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 4 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $Q = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 4 & 0 & 1 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. □

Problem 2: Let $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

(1) (3 points) Find the minimal polynomial of A.

Solution. $(t-1)(t+2)$ □

(2) (3 points) Explain why A is diagonalizable or not diagonalizable.

Solution. $\lambda_1 = 1, \lambda_2 = -2$.
Diagonalizable. G.M. of λ_1 = A.M. of $\lambda_1 = 1$ and G.M. of $\lambda_2 = 2$ =A.M. of $\lambda_2 = 2$. □

(3) (3 points) Find a Jordan canonical form J of A.

Solution. $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. ☐

(4) (5 points) Find a matrix Q such that $Q^{-1}AQ = J$.

Solution. $Q = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$. ☐

Problem 3: (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^3 = A$. Show that A is diagonalizable.

Proof. Let $g(t) = t^3 - t = t(t+1)(t-1)$. Then $g(A) = O$, and hence the minimal polynomial $p(t)$ of A divides $g(t)$. This also implies that A has only possible eigenvalues 0, -1 or 1. Since $g(t)$ has no repeated factors, neither does $p(t)$. Thus A is diagonalizable by Theorem 7.16. ☐

Problem 4: Let A be a square matrix with minimal polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(1) (3 points) A is invertible if and only if $a_0 \neq 0$.

Proof. Let $f(t)$ be the characteristic polynomial of A.
Then A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow f(0) \neq 0 \Leftrightarrow p(0) = a_0 \neq 0$. ☐

(2) (4 points) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right).$$

Proof. By the definition of minimal polynomial, we have

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0.$$

Hence

$$\begin{aligned} &A\left\{(-1/a_0)[A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n]\right\} \\ &\quad = (-1/a_0)[A^n + a_{n-1}A^{n-1} + \cdots + a_1 A] \\ &\quad = (-1/a_0)(-a_0 I_n) = I_n. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\left\{(-1/a_0)[A^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1I_n]\right\}A\\ =(-1/a_0)[A^n+a_{n-1}A^{n-1}+\cdots+a_1A]\\ =(-a_0I_n)(-1/a_0)=I_n.\end{aligned}$$

Therefore,

$$A^{-1}=(-1/a_0)[A^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1I_n].$$

□

Problem 5: (5 points) Let $V=P(\mathbb{R})$ with the inner product $< f(x), g(x) >= \int_0^2 f(t)g(t)dt$, and consider the subspace $P_2(\mathbb{R})$ with the standard ordered basis $\beta=\{1,x,x^2\}$. Use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1,v_2,v_3\}$ for $P_2(\mathbb{R})$,

Solution. $\left\{1, x-1, x^2-2x+\frac{2}{3}\right\}$. □

Problem 6: Let $T\in L(V)$, $V=\mathbb{C}^2$ and $F=\mathbb{C}$ such that

$$T(a_1,a_2)=(2ia_1+3a_2, 4a_1-ia_2).$$

(1) (4 points) Let $\beta=\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}$ be an ordered basis for $\mathbb{C}^2$. Compute $[T^*]_\beta$.

Solution. $<(a_1,a_2), T^*(2,1)>=<T(a_1,a_2),(2,1)>=\cdots=<(a_1,a_2),(-4i+4,6+i)>$. $\Rightarrow T^*(2,1)=(-4i+4,6+i)$. Similarly, $T^*(1,0)=(-2i,3)$. It is easy to show that $[T^*]_\beta=\begin{pmatrix}6+i & 3\\ -8-6i & -6-2i\end{pmatrix}$. □

(2) (4 points) Determine whether T is normal, self-adjoint, or neither.

Solution. Neither. □