NAME:____________ ID NO.:______________________ CLASS: __________

Problem 1: Let V be an inner product space, and let T be a linear operator on V. Prove the following resluts

(1) (5 points) $R(T^*)^\perp = N(T)$.

Proof. If $x \in R(T^*)^\perp$, then $0 =< x, T^*(y) >=< T(x), y >$ for any $y \in V$. This implies that $T(x) = 0$, i.e. $x \in N(T)$. Hence $R(T^*)^\perp \subseteq N(T)$.

If $x \in N(T)$, then $0 =< 0, y >=< T(x), y >=< x, T^*(y) >$ for any $y \in V$. This implies that $x \in R(T^*)^\perp$. Hence $N(T) \subseteq R(T^*)^\perp$.
Therefore, we conclude that $R(T^*)^\perp = N(T)$. □

(2) (1 point) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$. (Hint: Use the fact that if W is a subspace of a finite-dimensional inner product space V, then $W = (W^\perp)^\perp$.)

Proof. By hint, $N(T)^\perp = \left(R(T^*)^\perp\right)^\perp = R(T^*)$. □

(3) (4 points) If V is finite-dimensional, then $N(T^*T) = N(T)$. Deduce that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.

Proof. If $x \in N(T^*T)$, then $T^*T(x) = 0$ and $0 =< T^*T(x), x >=< T(x), T(x) >$. This implies that $T(x) = 0$, i.e. $x \in N(T)$. Hence $N(T^*T) \subseteq N(T)$. Conversely, if $x \in N(T)$, then $T^*T(x) = T^*(0) = 0$. This implies that $x \in N(T^*T)$. Hence $N(T) \subseteq N(T^*T)$. We conclude that $N(T) = N(T^*T)$. Finally, by dimension theorem, we have $\dim V = \dim N(T) + \dim \operatorname{rank}(T) = \dim N(T^*T) + \dim \operatorname{rank}(T^*T)$. Therefore, $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$. □

(4) (3 points) If V is finite-dimensional, then $\operatorname{rank}(T) = \operatorname{rank}(T^*)$.

Proof. By theorem, we have $\dim V = \dim N(T) + \dim N(T)^\perp$. By dimension theorem, we know that $\dim V = \dim N(T) + \dim R(T)$. Hence $\dim N(T)^\perp = \dim R(T)$. Also, we know that, by (b), $\dim N(T)^\perp = \dim R(T^*)$. Therefore, $\dim R(T) = \dim R(T^*)$, i.e. $\operatorname{rank}(T) = \operatorname{rank}(T^*)$. □

Problem 2: Give an example of a linear operator T on $\mathbb{R}^2$ and an ordered basis β for $\mathbb{R}^2$ such that T is normal, but $[T]_\beta$ is not normal.

(1) (3 points) Write down your example.

Solution. For example, let $T(a, b) = (a, 2b)$ and $\beta = \{(1, 1), (0, 1)\}$. □

(2) (3 points) Show that the T in your example is normal.

Solution. Check that T is self-adjoint. Hence T is normal. □

(3) (3 points) Show that the $[T]_\beta$ in your example is not normal.

Solution. $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. □

Problem 3: Let $T \in L(V)$, $V = \mathbb{C}^2$ and $F = \mathbb{C}$ such that

$$T(a_1, a_2) = (3ia_1 + 4a_2, 2a_1 - a_2).$$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be an ordered basis for $\mathbb{C}^2$.

(1) (3 points) Compute $([T]_\beta)^*$.

solution. $\begin{pmatrix} -3i + 8 & 6i - 16 \\ 4 & -9 \end{pmatrix}$. □

(2) (4 points) Compute $[T^*]_\beta$.

Solution. $< (a_1, a_2), T^*(1, 2) >=< T(a_1, a_2), (1, 2) >= \cdots =< (a_1, a_2), (-3i + 4, 2) >$. $\Rightarrow T^*(1, 2) = (-3i + 4, 2)$. Similarly, $T^*(0, 1) = (2, -1)$. It is easy to show that $[T^*]_\beta = \begin{pmatrix} -3i + 4 & 2 \\ 6i - 6 & -5 \end{pmatrix}$. □

Problem 4: (4 points) Prove that a 3×3 matrix that is both unitary and upper triangular must be a diagonal matrix.

Proof. It can be proved by straightforward computation. We skip the details. □

Problem 5:(5 points) Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\mathrm{tr}(A^* A) = \sum_{i=1}^{n} |\lambda_i|^2,$$

where the λ_i's are the (not necessarily distinct) eigenvalues of A.

Proof. By Theorem 6.19 and Theorem 6.20, A is similar to a diagonal matrix whose diagonal entries consist of eigenvalues, i.e. there exists an invertible matrix Q such that $Q^{-1}AQ = D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Hence $\mathrm{tr}(A^*A) = \mathrm{tr}((QDQ^{-1})^*(QDQ^{-1})) = \mathrm{tr}(QD^*DQ^*) = \mathrm{tr}(D^*D) = \sum_{i=1}^{n} |\lambda_i|^2$. □

Problem 6: Let $V = P(\mathbb{R})$ with the inner product $< f(x), g(x) >= \int_{-1}^{1} f(t)g(t)dt$, and consider the subspace $P_2(\mathbb{R})$ with the standard ordered basis $\beta = \{1, x, x^2\}$.

(1) (5 points) Use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$,

Solution. $\{1, x, x^2 - 1/3\}$. □

(2) (3 points) Use the orthogonal basis in (1) to obtain an orthonormal basis for $P_2(\mathbb{R})$.

Solution. $\left\{1/\sqrt{2}, \sqrt{3/2}x, \sqrt{5/8}(3x^2 - 1)\right\}$. □

Problem 7:(4 points) Let V be a finite dimensional inner product space, and X and Y be two subspaces of V. If $\dim X < \dim Y$, show that there exists a nonzero vector y in Y such that y is orthogonal to all vectors in X.

Proof. If y is Y such that y is orthogonal to all vectors in X, then $y \in X^\perp \cap Y$. So it is enough to show that $\dim(X^\perp \cap Y) > 0$. We know that $X^\perp + Y$ is a subspace of V, so $\dim(X^\perp + Y) \leq \dim V$. We also know that $\dim V = \dim X + \dim X^\perp$ and $\dim(X^\perp + Y) = \dim X^\perp + \dim Y - \dim(X^\perp \cap Y)$. Hence, by assumption and the above equalities, we have

$$\begin{aligned} \dim(X^\perp \cap Y) &= \dim X^\perp + \dim Y - \dim(X^\perp + Y) \\ &= (\dim V - \dim X) + \dim Y - \dim(X^\perp + Y) \\ &\geq \dim Y - \dim X > 0. \end{aligned}$$

□