5.1.6. Let λ be an eigenvalue of T corresponding to the eigenvector v. Then

$$Tv = \lambda v \Leftrightarrow [Tv]_\beta = [\lambda v]_\beta \Leftrightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta.$$

□

5.1.7 (a). Since $[T]_\beta = Q^{-1}[T]_\gamma Q$, where $Q = [I]_\beta^\gamma$. We have

$$\det([T]_\beta) = \det(Q^{-1}[T]_\gamma Q) = \det(Q^{-1})\det([T]_\gamma)\det(Q) = \det([T]_\gamma).$$

□

5.1.8 (a). T is invertible $\Leftrightarrow \det(T) \neq 0 \Leftrightarrow N(T) = \{0\}$. $\overset{Thm5.4}{\Longleftrightarrow}$ 0 is not an eigenvalue of T. □

5.1.8 (b). Suppose that the nonzero scalar λ is an eigenvalue of T.$\Leftrightarrow Tv = \lambda v \Leftrightarrow v = T^{-1}Tv = T^{-1}(\lambda v) = \lambda T^{-1}v$. $\Leftrightarrow \lambda^{-1}v = T^{-1}v$. □

Hint for 5.1.8 (c). Simply replace linear operator T by matrix M. □

Hint for 5.1.9. The determinant of an upper triangular matrix M are the product of the diagonal entries of M. □

5.1.11 (a). Assume that the square matrix A is similar to the scalar matrix λI. Hence $A = Q^{-1}(\lambda I)Q = \lambda I$. □

5.1.11 (b). Let M be an $n \times n$ diagonalizable matrix having only one eigenvalue λ. By theorem 5.1, there exists an ordered basis $\beta = \{v_1, v_2, \cdots, v_n\}$for F^n consisting of eigenvectors of T. Then $Mv_i = \lambda v_i$ for all $i = 1, 2, \cdots, n$. Hence $Mv = \lambda v$ for all $v \in \text{span}(\beta) = F^n$. So M must be λI. □

5.1.11(c). It is easy to see that 1 is the only eigenvalue for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not a scalar matrix, by (b) it is not diagonalizable. □

5.1.12(a). Use the fact that $\det(Q^{-1}AQ - \lambda I) = \det(Q^{-1}(A - \lambda I)Q) = \det(Q^{-1})\det(A - \lambda I)\det(Q) = \det(A - \lambda I)$. □

Hint for 5.1.12(b). Use (a) and the fact that matrix representations of a linear operator are similar to each other with respect to different choices of bases for the vector space. □

5.1.14. Use the fact that

$$\det(A - \lambda I) = \det((A - \lambda)^t) = \det(A^t - \lambda I).$$

□

5.1.15(a). Since $T^m x = T^{m-1}Tx = T^{m-1}(\lambda x) = \lambda T^{m-1}x = \cdots = \lambda^m x$. □

5.1.15(b). Simply replace the linear operator T by the matrix M in the statement and the proof. □

5.1.16(a). Let A be a square matrix, then, by exercise 2.3.13, $\operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(QQ^{-1}A) = \operatorname{tr}(A)$. for some invertible matrix Q of the same size as A. □

5.1.16(b). We may define the trace of a linear operator on a finite-dimensional vector space to be the trace of its matrix representation. It is well-defined, since the matrix representations of a linear operator on a finite-dimensional vector space are similar to each other with respect to different ordered bases chosen for the vector space. □

5.1.18(a). Since

$$\det(A + cB) = \det(B(B^{-1}A + cI)) = \det(B)\det(B^{-1}A + cI).$$

Hence $\det(B) \neq 0$ implies that $\det(A+cB) = 0 \Leftrightarrow \det(B^{-1}A+cI) = 0$. Since $\det(B^{-1}A+cI)$ is a polynomial of c with complex coefficients, we can always find some scalar $c \in \mathbb{C}$ so that $\det(B^{-1}A + cI) = 0$. Hence there exist a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible. □

5.1.18(b). Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $\det(A) = \det(A + cB) = 1$. Hence both A and $A + cB$ are invertible for all $c \in \mathbb{C}$. □

5.1.20. By assumption and definition $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = \det(A - tI)$. Hence $f(0) = a_0 = \det(A)$. □