5.2.8:

Proof. Since $\dim E_{\lambda_2} \geq 1$, we can choose a nonzero vector $v \in E_{\lambda_2}$. Let $\beta = \{v_1, v_2, \cdots, v_{n-1}\}$ be a basis for E_{λ_1}. Then, by Theorem 5.8, $\{v, v_1, v_2, \cdots, v_{n-1}\}$ forms a basis for F^n consisting of eigenvectors of A. Hence, Theorem 5.1 implies that A is diagonalizable. □

5.2.9:

Proof of (a). Since the characteristic polynomial of T is independent of the choice of the ordered basis β and the matrix $[T]_\beta - tI$ is an upper triangular matrix, the characteristic polynomial

$$f(t) = \det([T]_\beta - tI) = \prod_{i=1}^{n} \big(([T]_\beta)_{ii} - t\big)$$

splits. □

Statement of (b). Suppose that $A \in M_{n\times n}(F)$ is similar to an upper triangular matrix B. Prove that the characteristic polynomial for A splits. □

Proof of (b). Since A is similar to B, the characteristic polynomial for A is the same as the characteristic polynomial for B. Also since B is an upper triangular matrix, the characteristic polynomial

$$f(t) = \det(A - tI) = \det(B - tI) = \prod_{i=1}^{n} (B_{ii} - t)$$

splits. □

5.2.10:

Proof. By 5.2.9(a), the characteristic polynomial

$$f(t) = \det([T]_\beta - tI) = \prod_{i=1}^{n} \big(([T]_\beta)_{ii} - t\big)$$

implies that the diagonal entries of $[T]_\beta$ are eigenvalues of T. Then the result follows from the assumption. □

5.2.11:

Proof. Similar matrices have the same characteristic polynomial and, hence, the same eigenvalues. We also know that similar matrices have the same trace and the same determinant. Then 5.2.10 and assumptions imply that $\mathrm{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$ and $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$. □

5.2.12:

Proof of (a). Let E_λ be the eigenspace of the invertible operator T corresponding to the eigenvalue λ and $E_{\lambda^{-1}}$ be the eigenspace of the invertible operator T corresponding to the eigenvalue λ^{-1}. Let $v \in E_\lambda$, then

$$T(v) = \lambda v \Rightarrow v = T^{-1}(\lambda v) = \lambda T^{-1}(v) \Rightarrow T^{-1}(v) = \lambda^{-1} v.$$

This implies that $v \in E_{\lambda^{-1}}$. Similarly, let $v \in E_{\lambda^{-1}}$, then $v \in E_\lambda$. Therefore $E_\lambda = E_{\lambda^{-1}}$. □

Proof of (b). By (a) and Theorem 5.1, if the invertible operator T is diagonalizable, then the basis consisting of eigenvectors of T will also be the basis consisting of eigenvectors of T^{-1}. Therefore, Theorem 5.1 implies that T^{-1} is diagonalizable. □

5.2.13:

Proof of (a). The matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has eigenvalues 0 and 1. For the eigenvalue 0, $E_0 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is the eigenspace for A and $E_0' = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is the eigenspace for A^t. □

Proof of (b). Since

$$\dim(E_\lambda) = \operatorname{nullity}(A - \lambda I) = \operatorname{nullity}((A - \lambda I)^t) = \operatorname{nullity}(A^t - \lambda I) = \dim(E_\lambda').$$

□

Proof of (c). If A is diagonalizable, then its characteristic polynomial splits and for each eigenvalue λ the multiplicity of λ is the same as $\dim(E_\lambda)$. Since A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities, then the characteristic polynomial of A^t splits and, by (b), for each eigenvalue λ the multiplicity of λ is the same as $\dim(E_\lambda')$. Therefore A^t is diagonalizable. □

5.2.18:

Proof of (a). Let β be the ordered basis for V such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Since $[T]_\beta$ and $[U]_\beta$ are diagonal matrices, $[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$. Therefore T and U commute. □

Proof of (b). Let Q be the invertible matrix such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. Then we have

$$\left(Q^{-1}AQ\right)\left(Q^{-1}BQ\right) = \left(Q^{-1}BQ\right)\left(Q^{-1}AQ\right) \Rightarrow Q^{-1}ABQ = Q^{-1}BAQ \Rightarrow AB = BA.$$

□

5.2.19:

Proof. By 5.1.15(a), T and T^m have the same eigenvectors. Hence T and T^m are simultaneously diagonalizable. □